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Final Coalgebras in Accessible Categories

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We propose a construction of the final coalgebra for a finitary endofunctor of a finitely accessible category and study conditions under which this construction is available. Our conditions always apply when the accessible category is cocomplete, hence is a locally finitely presentable (l.f.p.) category. Thus we give an explicit and uniform construction of the final coalgebra in this case. On the other hand, there are interesting examples of final coalgebras beyond the realm of l.f.p. categories to which our results apply. In particular we construct the final coalgebra for every finitary endofunctor on the category of linear orders and we analyze Freyd's coalgebraic characterization of the closed unit as an instance of this construction. We use and extend results of Tom Leinster, developed for his study of self-similar objects in topology, relying heavily on his formalism of modules (corresponding to endofunctors) and complexes for a module.

1. Introduction

Coalgebras for an endofunctor (of, say, the category of sets) are well-known to describe *systems of formal recursive equations*. Such a system of equations then specifies a potentially infinite "computation" and one is naturally interested in giving (uninterpreted) semantics to such a computation. In fact, such semantics can be given by means of a coalgebra again: this time by the *final* coalgebra for the given endofunctor.

Let us give a simple example of that.

Example 1.1. Suppose that we fix a set A and we want to consider the set A^ω of infinite sequences of elements of A, called *streams*. Moreover, we want to define a function $\mathsf{zip} : A^\omega \times A^\omega \longrightarrow A^\omega$ that "zips up" two streams, i.e., the equality

$$\mathsf{zip}\Big((a_0, a_1, a_2, \ldots), (b_0, b_1, b_2, \ldots)\Big) = (a_0, b_0, a_1, b_1, a_2, b_2, \ldots)$$

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holds.

One possible way of working with infinite expressions like streams is to introduce an additional approximation structure on the set of infinite expressions and to speak of an infinite expression as of a "limit" of its finite approximations, either in the sense of a complete partial order or of a complete metric space, see (Goguen, Thatcher, Wagner and Wright (1977)) and (America and Rutten (1989)), respectively. Such an approach may get rather technical and the additional approximation structure may seem rather arbitrary.

In fact, using the ideas of Calvin Elgot and his collaborators, see (Elgot (1975)) and (Elgot, Bloom and Tindell (1978)), combined with a coalgebraic approach to systems of recursive equations (Rutten (2000)) and (Aczel, Adámek, Milius and Velebil (2003)), one may drop the additional structure altogether and define solutions by *corecursion*, i.e., by means of a final coalgebra.

Clearly, the above zipping function can be specified by a *system of recursive equations*

$$\mathsf{zip}(a,b) = (\mathsf{head}(a), \mathsf{zip}(b, \mathsf{tail}(a))) \tag{1.1}$$

one equation for each pair a, b of streams, where we have used the functions $\mathsf{head}(a_0, a_1, a_2, \ldots) = a_0$ and $\mathsf{tail}(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$.

In fact, the above system (1.1) of recursive equations can be encoded as a map

$$e : A^\omega \times A^\omega \longrightarrow A \times A^\omega \times A^\omega, \quad (a,b) \mapsto (\mathsf{head}(a), b, \mathsf{tail}(a)) \tag{1.2}$$

This means that we rewrote the system (1.1) as a coalgebra and we will show now that a final coalgebra gives its unique solution, namely the function zip. To this end, we define first an endofunctor Φ of the category of sets by the assignment

$$X \mapsto A \times X$$

A *coalgebra* for Φ (with an *underlying set* X) is then any mapping $e : X \longrightarrow \Phi X$, i.e., a mapping of the form

$$e : X \longrightarrow A \times X$$

Suppose that a *final coalgebra*

$$\tau : TA \longrightarrow A \times TA$$

for Φ exists. Its finality means that for *any* coalgebra $c : Z \longrightarrow A \times Z$ there exists a unique mapping $c^\dagger : Z \longrightarrow TA$ such that the square

$$\begin{array}{ccc} Z & \xrightarrow{\;c\;} & A \times Z \\ {\scriptstyle c^\dagger}\downarrow & & \downarrow{\scriptstyle A \times c^\dagger} \\ TA & \xrightarrow[\;\tau\;]{} & A \times TA \end{array} \tag{1.3}$$

commutes. Moreover, it is well-known that the mapping τ must be a bijection due to finality. Luckily, in our case the final coalgebra is well-known to exist and has the following description: TA is the set of all streams A^ω and the mapping τ sends $a \in A^\omega$ to the pair $(\mathsf{head}(a), \mathsf{tail}(a))$.

If we instantiate the coalgebra e from (1.2) for c in the above square and if we chase the elements of $A^\omega \times A^\omega$ around it, we see that the uniquely determined function $e^\dagger : A^\omega \times A^\omega \longrightarrow A^\omega$ satisfies the recursive equation (1.1).

The reason for the existence of a final coalgebra for Φ is that both the category of sets and the endofunctor Φ are "good enough": the category of sets is locally finitely presentable and the functor is finitary (we explain what that means in more detail below).

However, it is not the case that a final coalgebra exists for every "good enough" functor: for example the identity endofunctor of the category of sets and injections does not have a final coalgebra for cardinality reasons. Yet there are examples of interesting endofunctors of "less good" categories that still have a final coalgebra, see, e.g., Example 4.1 below.

On the other hand the need to look for final coalgebras in environments different that of Set and all functions as arrows is indisputable: In the above example choosing $A = \mathbb{N}$, one obtains $\mathbb{N}^\mathbb{N}$ as the final coalgebra and that is well known to be isomorphic (in Set) to the continuum. As (Pavlović and Pratt (2002)) argue though, it is not possible to reveal the order of the continuum out of this characterization of the underlying set as a final coalgebra. In order to achieve that, one has to introduce structure on Set, for example by working in the category Pos, the category of partially ordered sets and order-preserving maps.

Structured (possibly many-sorted) sets satisfying properties that are expressed syntactically by sentences of a certain reasonable complexity (and homomorphisms between them) are organized in *locally finitely presentable categories (l.f.p)*. This appears to many people to be the right level of generality for discussing (final) coalgebras. See (Adámek (2003)), (Klin (2007)), (Breugel, Hermida, Makkai and Worrell (2007)), (Worrell (2005)). It is well known that final coalgebras exist for accessible endofunctors (= determined by small pieces of data) on locally presentable categories. This is due to the fact that categories of coalgebras are a certain kind of 2-categorical limit, and the category of locally presentable categories with accessible functors as 1-morphisms are closed under such limits (this is essentially due to (Makkai and Paré (1989)), but see also (Breugel, Hermida, Makkai and Worrell (2007))). On the other hand there is no hitherto known *construction* that applies to all locally (finitely) presentable categories and (finitely) accessible endofunctors. Such constructions are known for the category of sets and finitary endofunctors ((Worrell (2005))) but the attempts to extend these constructions to locally finitely presentable categories had only partial success ((Adámek (2003))). We supply here such a construction. The important thing, however, is that our uniform description of final coalgebras will be very reminiscent of streams: the coalgebra structure of a final coalgebra is *always* given by analogues of head and tail mappings from the previous example.

Dealing with l.f.p categories may itself not be sufficient. Many structures that enjoy properties of bigger syntactical complexity (e.g fields, linear orders, orders with specified endpoints) are organized in (what is more general than l.f.p) finitely accessible categories. We give here conditions that ensure the existence of final coalgebras in this wider context. One may object here the possible utility of such an endeavor: rather than studying the final coalgebra in an environment of structured sets with rich properties (e.g linear orders), specify the final coalgebra in an environment of structured sets with less rich properties (e.g posets) and verify that the particular object has the richer properties (is linearly ordered). This objection would be completely justified if we knew in advance the object and tried to describe it as a final coalgebra (as it is the case when trying to describe the continuum, the Cantor set or the Baire space as a final coalgebra (Pavlović and Pratt (2002))). But for a general accessible endofunctor, say on an l.f.p category, identifying the final coalgebra as a known object (notwithstanding the fact that we supply a construction of it), may involve some guessing. In contrast, the proposed sufficient conditions may be easy to verify. For example we show that every finitary endofunctor on the category of linear orders and order-preserving maps admits a final coalgebra (see Corollary 5.17). Furthermore, Freyd's characterization of the closed unit interval as a final coalgebra ((Freyd (1999)), (Freyd (2008))) shows that the interesting contexts to discuss final coalgebras go beyond the l.f.p case.

The organization of the paper

In this work we will make advantage of the fact that finitary endofunctors of finitely accessible categories can be fully reconstructed from *essentially small data*. In fact, finitary endofunctors can be replaced by *flat modules* on the *small categories of finitely presentable objects*. Such pairs

(small category, flat module)

fully encode the pattern of the recursive process in question.

We recall the concepts of finitary functors and finitely accessible categories and the process of passing from endofunctors to modules in Section 2.

In Section 3 we indicate how complexes emerge when one tries to prove a fixed point lemma for finitary endofunctors on l.f.p categories, imitating the proof of the classical Knaster-Tarski lemma. We introduce there the main tool of the paper — the *category of complexes* for a (flat) module. The category of complexes will then allow us to give a concrete description of final coalgebras. Furthermore, we formulate a condition on the category of complexes that ensures that a final coalgebra for the module in question exists, see Theorem 3.10 below. As a byproduct we obtain, in Corollary 3.12, a new proof of the well-known fact that every finitary endofunctor of a *locally finitely presentable category* has a final coalgebra. Moreover, we prove that the elements of the final coalgebra are essentially the complexes.

In Section 4 we discuss endofunctors on the category of partially ordered sets with distinct endpoints. This category is finitely accessible but not l.f.p. We explain how our conditions are applied to guarantee that the final coalgebra exists for large classes of endofunctors on this category (and possibly similar ones). Freyd, in characterizing the closed unit interval as a final coalgebra, works exactly in this category. We show how our description of the final coalgebra yields the closed unit interval in this case. Finally, and in the same vein, we indicate how our construction guarantees the existence of cofree coalgebras for every finitary endofunctor on such a category.

Although the results of Section 3 give a concrete desription of the final coalgebra in the more general framework of finitely accessible categories, the condition (given there) which ensures that the underlying object of the final coalgebra lives in the category, is rather hard to verify. We devote Section 5 to a certain weakening of this condition. The weaker condition on the category of complexes of the module yields a final coalgebra as well but the module has to satisfy a certain side condition of finiteness flavor. We also elaborate on that finiteness condition so that, as said above, we can verify that it is satisfied by all finitary endofunctors on linear orders.

In some cases, one can prove that the conditions we give are *necessary and sufficient* for the existence of a final coalgebra. We devote Section 6 to finding conditions on the endofunctor that ensure the existence of such a characterization. The necessity of our conditions applies to endofunctors that are *pointed*, i.e they admit a natural transformation from the identity functor. Endofunctors that arise as composites of adjoints form examples.

Related work

This work is very much influenced by the work of Tom Leinster, (Leinster (2011)) on self-similarity in topology, as it builds on his ideas of a complex for a module.

In fact, Leinster works with categories that are “accessible” for the notion of componentwise filtered, i.e with categories of functors that satisfy the flatness conditions of Section 2 only with respect to finite connected diagrams. Our contribution in Section 3 lies in the fact that we show that his methods can be applied as to cover the case of general flat functors. Put differently, but in a technically equivalent manner, Leinster works with flat functors on categories that arise as cocompletions of small categories under coproducts. Extending his methods to the study of flat functors on all small categories allows us to account for finitary functors on l.f.p categories. Thus his method of construction of the final coalgebra turns out to yield a general method for describing final coalgebras for finitary functors on l.f.p categories. A modified notion of complex is used in (Karazeris, Matzaris and Velebil) (work in progress) for constructing cofree coalgebras for finitary endofunctors on finitely accessible categories.

Leinster provides, in his framework, sufficient weak conditions for the existence of the final coalgebra. They involve a finiteness condition as well. His finiteness condition is too restrictive in our framework. Our Section 5 gives a different perspective than his and involves different methods.

Other descriptions of final coalgebras, that work in more restrictive frameworks, follow from the analysis of the final coalgebra sequence, see (Adámek (2003)), (Worrell (2005)). We indicate how our construction relates to that.

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2. Preliminaries

In this preliminary section we introduce the notation and terminology that we will use in the rest of the paper. Most of it is fairly standard, we refer to books (Adámek and Rosický (1994)) and (Borceux (1994)) for the material concerning finitely accessible categories and finitary functors.

Coalgebras and final coalgebras

We give a precise definition of (final) coalgebras, see, e.g., (Rutten (2000)) for motivation and examples of various coalgebras in the category of sets.

Definition 2.1. Suppose $\Phi : \mathscr{K} \longrightarrow \mathscr{K}$ is any functor.

(1) A *coalgebra for* Φ is a morphism $e : X \longrightarrow \Phi(X)$.

(2) A *homomorphism of coalgebras* from $e : X \longrightarrow \Phi(X)$ to $e' : X' \longrightarrow \Phi(X')$ is a morphism $h : X \longrightarrow X'$ making the following square

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & \Phi(X) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Phi(h)} \\ X' & \xrightarrow[\;e'\;]{} & \Phi(X') \end{array}$$

commutative.

(3) A coalgebra $\tau : T \longrightarrow \Phi(T)$ is called *final*, if it is a terminal object of the category of coalgebras, i.e., if for every coalgebra $e : X \longrightarrow \Phi(X)$ there is a unique morphism $e^\dagger : X \longrightarrow T$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & \Phi(X) \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle \Phi(e^\dagger)} \\ T & \xrightarrow[\;\tau\;]{} & \Phi(T) \end{array}$$

commutes.

Finitely accessible and locally finitely presentable categories

Finitely accessible and locally finitely presentable categories are those where every object can be reconstructed knowing its "finite parts". This property has, for example, the category Set of sets and mappings, where a set P is recognized as finite exactly when its hom-functor $\mathsf{Set}(P, -) : \mathsf{Set} \longrightarrow \mathsf{Set}$ preserves colimits of a certain class — the so-called *filtered* colimits.

A colimit of a general diagram $D : \mathscr{D} \longrightarrow \mathscr{K}$ is called *filtered*, provided that its scheme-category $\mathscr{D}$ is filtered. A category $\mathscr{D}$ is called *filtered* provided that every finite subcategory of $\mathscr{D}$ admits a cocone. In more elementary terms, filteredness of $\mathscr{D}$ can be expressed equivalently by the following three properties:

(1) The category $\mathscr{D}$ is nonempty.

(2) Each pair d_1, d_2 of objects of $\mathscr{D}$ has an "upper bound", i.e., there exists a cocone

$$\begin{array}{ccc} d_1 & & \\ & \searrow & \\ & & d \\ & \nearrow & \\ d_2 & & \end{array}$$

in $\mathscr{D}$.

(3) Each parallel pair of morphisms in $\mathscr{D}$ can be "coequalized", i.e., for each parallel pair

$$d_1 \rightrightarrows d_2$$

of morphisms in $\mathscr{D}$ there is a completion to a commutative diagram of the form

$$d_1 \rightrightarrows d_2 \dashrightarrow d$$

in $\mathscr{D}$.

A category is $\mathscr{D}$ called *cofiltered* provided that the dual category $\mathscr{D}^{op}$ is filtered.

An object P of a category $\mathscr{K}$ is called *finitely presentable* if the hom-functor $\mathscr{K}(P, -) : \mathscr{K} \longrightarrow \mathsf{Set}$ preserves filtered colimits.

Definition 2.2. A category $\mathscr{K}$ is called *finitely accessible* if it has filtered colimits and if it contains a small subcategory consisting of finitely presentable objects such that every object of $\mathscr{K}$ is a filtered colimit of these finitely presentable objects.

A cocomplete finitely accessible category is called *locally finitely presentable*.

Remark 2.3. Locally finitely presentable categories were introduced by Peter Gabriel and Friedrich Ulmer (Gabriel and Ulmer (1971)), finitely accessible categories were introduced by Christian Lair (Lair (1981)) under the name *sketchable* categories. Tight connections of these concepts to (infinitary) logic can be found in the book (Makkai and Paré (1989)), the book (Adámek and Rosický (1994)) deals with the connection of these concepts to categories of structures.

Example 2.4.

(1) The category Set of sets and mappings is locally finitely presentable. The finitely presentable objects are exactly the finite sets.

(2) Every variety of finitary algebras is a locally finitely presentable category. The finitely presentable objects are exactly the algebras that are presented by finitely many generators and finitely many equations in the sense of universal algebra.

(3) The category Inj having sets as objects and injective maps as morphisms is a finitely accessible category that is not locally finitely presentable. The finitely presentable objects are exactly the finite sets.

(4) Denote by Field the category of fields and field homomorphisms. Then Field is a finitely accessible category that is not locally finitely presentable.

(5) The category Lin of linear orders and monotone maps is finitely accessible but not locally finitely presentable. The finitely presentable objects are exactly the finite ordinals.

(6) Let $\mathsf{Pos}_{0,1}$ denote the following category:

(a) Objects are posets having distinct top and bottom elements.

(b) Morphisms are monotone maps preserving top and bottom elements.

Then $\mathsf{Pos}_{0,1}$ is a *Scott complete* category in the sense of Jiří Adámek (Adámek (1997)): it is finitely accessible and every small diagram in $\mathsf{Pos}_{0,1}$ that has a cocone, has a colimit.

Scott complete categories are therefore "not far away" from being cocomplete and thus locally finitely presentable.

However, $\mathsf{Pos}_{0,1}$ is not locally finitely presentable since it lacks a terminal object. Finitely presentable objects in $\mathsf{Pos}_{0,1}$ are exactly the finite posets having distinct bottom and top elements.

(7) The category of topological spaces and continuous maps is not finitely accessible. Although this category has filtered (in fact, all) colimits, the only finitely presentable objects are finite discrete topological spaces and these do not suffice for reconstruction of a general topological space.

Of course, more examples of "everyday-life" finitely accessible categories can be found in the literature, see, e.g., papers (Diers (1980)) and (Diers (1980)) by Yves Diers.

Flat functors

Every finitely accessible category $\mathscr{K}$ is equivalent to a category of the form

$$\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$$

(where $\mathscr{A}$ is a small category) that consists of all *flat functors* $X : \mathscr{A} \longrightarrow \mathsf{Set}$ and all natural transformations between them.

A functor $X : \mathscr{A} \longrightarrow \mathsf{Set}$ is called *flat* if its *category of elements* $\mathsf{elts}(X)$ is cofiltered. The category $\mathsf{elts}(X)$ has pairs (x, a) with $x \in Xa$ as objects and as morphisms from (x, a) to (x', a') those morphisms $f : a \longrightarrow a'$ in $\mathscr{A}$ with the property that $Xf(x) = x'$.

Flat functors X can be characterized by any of the following equivalent conditions:

(1) The functor $X : \mathscr{A} \longrightarrow \mathsf{Set}$ is a filtered colimit of representable functors.

(2) The left Kan extension $\mathrm{Lan}_Y X : [\mathscr{A}^{op}, \mathsf{Set}] \longrightarrow \mathsf{Set}$ of $X : \mathscr{A} \longrightarrow \mathsf{Set}$ along the Yoneda embedding $Y : \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathsf{Set}]$ preserves finite limits.

In case when $\mathscr{K}$ is locally finitely presentable one can prove that $\mathscr{K}$ is equivalent to the category

$$\mathsf{Lex}(\mathscr{A}, \mathsf{Set})$$

of all finite-limits-preserving functors on a small finitely complete category $\mathscr{A}$. In fact, the flat functors are exactly the finite-limits-preserving ones in this case.

Example 2.5. In this example we show how to express Set as a category of flat functors. Denote by $E : \mathsf{Set}_{fp} \longrightarrow \mathsf{Set}$ the full dense inclusion of an essentially small category of finite sets. In fact, in this example, we choose as a representative set of finitely presentable objects the set of finite ordinals.

The correspondence

$$X \mapsto \mathsf{Set}(E-, X)$$

then provides us with an equivalence

$$\mathsf{Set} \simeq \mathsf{Flat}(\mathsf{Set}_{\mathit{fp}}^{op}, \mathsf{Set}) = \mathsf{Lex}(\mathsf{Set}_{\mathit{fp}}^{op}, \mathsf{Set})$$

of categories. The slogan behind this correspondence is the following one:

> Instead of describing a set X by means of its elements $x \in X$ (as we do in Set), we describe a set by "generalized elements" of the form $n \longrightarrow X$, where n is a finite ordinal.

Thus, a set X now "varies in time": the hom-set $\mathsf{Set}(n, X)$ is the "value" of X at "time" n.

Remark 2.6. The above example is an instance of a general fact: every finitely accessible category $\mathscr{K}$ is equivalent to $\mathsf{Flat}(\mathscr{K}_{\mathit{fp}}^{op}, \mathsf{Set})$, where $E : \mathscr{K}_{\mathit{fp}} \longrightarrow \mathscr{K}$ denotes the full inclusion of the essentially small subcategory consisting of finitely presentable objects.

The equivalence works as follows: the flat functor $X : \mathscr{K}_{\mathit{fp}}^{op} \longrightarrow \mathsf{Set}$ is sent to the object

$$X \star E$$

which is a colimit of E weighted by X. Such a colimit is defined as an object $X \star E$ together with an isomorphism

$$\mathscr{K}(X \star E, Z) \cong [\mathscr{K}_{\mathit{fp}}^{op}, \mathsf{Set}](X, \mathscr{K}(E-, Z))$$

natural in Z. The above colimit can be considered to be an "ordinary" colimit of the diagram of elements of X:

$$x \in Xa \mapsto Ea$$

This explains the weight terminology: every Ea is going to be counted "Xa-many times" in the colimit $X \star E$. See (Borceux (1994)) for more details.

Flat modules

On finitely accessible categories there is class of functors that can be fully reconstructed by knowing their values on "finite parts". An example is the finite-powerset endofunctor

$$P_{\mathit{fin}} : X \mapsto \{S \mid S \subseteq X,\ S \text{ is finite }\}$$

of the category of sets. Such endofunctors can be characterized as exactly those *preserving* filtered colimits.

Definition 2.7. A functor $\Phi : \mathscr{K} \longrightarrow \mathscr{L}$ between finitely accessible categories is called *finitary* if it preserves filtered colimits.

By the above considerations, every finitary endofunctor $\Phi : \mathscr{K} \longrightarrow \mathscr{K}$ of a finitely accessible category $\mathscr{K}$ can be considered, to within equivalence, as a finitary endofunctor

$$\Phi : \mathsf{Flat}(\mathscr{A}, \mathsf{Set}) \longrightarrow \mathsf{Flat}(\mathscr{A}, \mathsf{Set})$$

Since the full embedding $\mathscr{A}^{op} \longrightarrow \mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ exhibits $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ as a free cocompletion of $\mathscr{A}^{op}$ w.r.t. filtered colimits (also denoted as $\mathrm{Ind}(\mathscr{A})$, the "inductive" cocompletion), we can then reconstruct Φ from a mere functor

$$M_\Phi : \mathscr{A}^{op} \longrightarrow \mathsf{Flat}(\mathscr{A}, \mathsf{Set})$$

(no preservation properties) by means of filtered colimits.

The latter functor can be identified with a functor of the form $M_\Phi : \mathscr{A}^{op} \times \mathscr{A} \longrightarrow \mathsf{Set}$ with the property that every $M_\Phi(a, -) : \mathscr{A} \longrightarrow \mathsf{Set}$ is flat. Such functors of two variables (without the extra flatness property) are commonly called *modules*. We will give the extra property a name.

Definition 2.8. A *module* $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!\!\mapsto\; \mathscr{B}$ from a small category $\mathscr{A}$ to a small category $\mathscr{B}$ is a functor $M : \mathscr{A}^{op} \times \mathscr{B} \longrightarrow \mathsf{Set}$. Given two such modules, M and N, a *module morphism* $M \longrightarrow N$ is a natural transformation between the respective functors.

A module M as above is called *flat* if every partial functor $M(a, -) : \mathscr{B} \longrightarrow \mathsf{Set}$ is a flat functor in the usual sense.

Remark 2.9. The above module terminology makes perfect sense if we denote an element $m \in M(a, b)$ by an arrow

$$a \xrightarrow{\;m\;}\!\!\!\!\!\!\!\!\!\!\!\!\;\; | \;\;\;\; b$$

and think of it as of a "vector" on which the categories $\mathscr{A}$ and $\mathscr{B}$ can act by means of their morphisms ("scalars"):

(1) Given $f : a' \longrightarrow a$ in $\mathscr{A}$, then

$$a' \xrightarrow{\;f\;} a \xrightarrow{\;m\;} b$$

denotes the element $M(f, b)(m) \in M(a', b)$.

Had we denoted such an action by $m@f$, then it is obvious that equations $m@(f \cdot f') = (m@f)@f'$ and $m@1_a = m$ hold — something that we know from classical module theory.

(2) Given $g : b \longrightarrow b'$ in $\mathscr{B}$, then

$$a \xrightarrow{\;m\;} b \xrightarrow{\;g\;} b'$$

denotes the element $M(a, g)(m) \in M(a, b')$.

(3) Functoriality of M gives an unambiguous meaning to diagrams of the form

$$a' \xrightarrow{\;f\;} a \xrightarrow{\;m\;} b \xrightarrow{\;g\;} b'$$

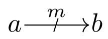
(4) We also extend the notion of commutative diagrams. For example, by saying that the following square

$$\begin{array}{ccc} a & \xrightarrow{\;m\;} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ a' & \xrightarrow[\;m'\;]{} & b' \end{array}$$

commutes we mean that the equality $m'@f = g@m$ holds.

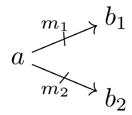
Remark 2.10. The broken arrow notation also allows us to formulate flatness of a module $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{B}$ in elementary terms. Namely, for every a in $\mathscr{A}$ the following three conditions must be satisfied:

(1) There is a broken arrow

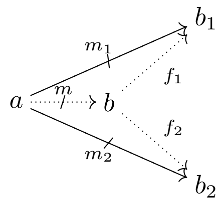


for some b in $\mathscr{B}$.

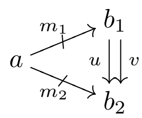
(2) For any two broken arrows



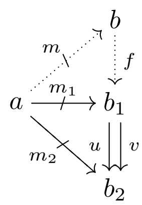
there is a commutative diagram



(3) For every commutative diagram



there is a commutative diagram



Example 2.11. In this example we show how the finitary endofunctor

$$X \mapsto X \times X + A$$

of the locally finitely presentable category Set can be viewed as a flat module.

In this sense, we identify the endofunctor $X \mapsto X \times X + A$ of Set with the endofunctor

$$\Phi : \mathsf{Set}(E-, X) \mapsto \mathsf{Set}(E-, X \times X) + \mathsf{Set}(E-, A)$$

of $\mathsf{Flat}(\mathsf{Set}_{fp}^{op}, \mathsf{Set})$. The corresponding flat module

$$M : \mathsf{Set}_{fp}^{op} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathsf{Set}_{fp}^{op}$$

then has values

$$M(a,b) = \mathsf{Set}_{fp}(b, a \times a) + \mathsf{Set}(b, A)$$

at finite ordinals a, b.

The above resemblance to classical module theory† can be pushed further: modules can composed by "tensoring" them.

Definition 2.12. Suppose $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{B}$ and $N : \mathscr{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{C}$ are modules. By

$$N \otimes M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{C}$$

we denote their *composition* which is defined objectwise by means of a coend

$$\Big(N \otimes M\Big)(a,c) = \int^{b} N(b,c) \times M(a,b)$$

Remark 2.13. A coend is a special kind of colimit. The elements of $\Big(N \otimes M\Big)(a,c)$ are equivalence classes. A typical element of $\Big(N \otimes M\Big)(a,c)$ is an equivalence class $[(n,m)]$ represented by a pair $(n,m) \in N(b,c) \times M(a,b)$ where the equivalence is generated by requiring the pairs

$$(n, f@m) \quad \text{and} \quad (n@f, m)$$

to be equivalent, where n, f and m are as follows:

$$a \xrightarrow{m} b \xrightarrow{f} b' \xrightarrow{n} c$$

Above, we denoted the actions of M and N by the same symbols, not to make the notation heavy.

It is well-known (see (Borceux (1994))) that the above composition organizes modules into a *bicategory*: the composition is associative only up to a coherent isomorphism and the *identity module* $\mathscr{A} : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{A}$, sending (a', a) to the hom-set $\mathscr{A}(a', a)$, serves as a unit only up to a coherent isomorphism. The following result is then easy to prove.

Lemma 2.14. Every identity module is flat and composition of flat modules is a flat module.

Remark 2.15. The above composition of modules makes one to attempt to draw diagrams such as

$$a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0$$

for elements $m_1 \in M(a_1, a_0)$, $m_2 \in M(a_2, a_1)$ of a module $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{A}$. Such diagrams are, however, to be considered only formally — *we never compose two "broken" arrows*.

† The resemblance can be made precise by passing to enriched category theory, see (Borceux (1994)).

The tensor notation from the above paragraphs allows us to pass from endofunctors to modules completely.

Observe that any flat functor $X : \mathscr{A} \longrightarrow \mathsf{Set}$ can be considered as a flat module $X : \mathbf{1} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{A}$ where $\mathbf{1}$ denotes the one-morphism category.

Then, given a flat module $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{A}$, the assignment $X \mapsto M \otimes X$ defines a finitary endofunctor of $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$.

In fact, every finitary endofunctor Φ of $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ arises in the above way: construct the flat module M_Φ as above, then there is an isomorphism

$$\Phi \cong M_\Phi \otimes -$$

of functors.

The category of complexes

Formal chains of "broken arrows" will be the main tool of the rest of the paper. We define a category of such chains (this definition comes from the paper (Leinster (2011)) of Tom Leinster).

Assumption 2.16. In the rest of the paper,

$$M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{A}$$

denotes a flat module on a small category $\mathscr{A}$. The pair $(\mathscr{A}, M)$ is called a *self-similarity system*.

Remark 2.17. The terminology *self-similarity system* is due to Tom Leinster (Leinster (2011)) and has its origin in the intention to study (topological) spaces that are self-similar. Since we refer to (Leinster (2011)) below, we keep the terminology, although our motivation is different.

3. Complexes and The Strong Solvability Condition

We present here our main technical tool, the category of complexes for a module. The notion of complex has been introduced by T. Leinster (Leinster (2011)), in the course of his study of self-similar objects. It encodes precisely the idea that a self-similar object is obtained via successive approximations. We adopt a slightly different approach in order to motivate this notion: Let us view coalgebras as generalizing the pre-fixed points of a continuous endofunctor of a poset and try to think of a "greatest fixed point lemma", a lá Knaster-Tarski.

Consider any finitely accessible category $\mathscr{K}$ and a finitary endofunctor $\Phi : \mathscr{K} \to \mathscr{K}$. We form the category of coalgebras, $\mathsf{Coalg}(\Phi)$, and try to find a terminal object in it. In the posetal case one would define it as

$$x_0 = \bigvee \{x \in \mathscr{K} \mid x \leq \Phi(x)\},$$

provided such a supremum existed in $\mathscr{K}$. By analogy one would wish to form the colimit over all coalgebras. This poses a size problem in the case of a category, but since $\mathscr{K}$ is finitely accessible we may try to restrict ourselves to coalgebras with finitely presented carriers. Since it is not the case now that all coalgebras participate in the colimit that defines the purported final coalgebra, we are faced with the problem of defining a coalgebra morphism out of an arbitrary coalgebra $(K, e : K \to \Phi(K))$, $K \in \mathscr{K}$. Since we know that $\mathscr{K}$ is finitely accessible and the functor Φ finitary we have:

— K is a filtered colimit of finitely presentable objects,

$$K = \operatorname{colim}(\mathscr{K}_{fp} \downarrow K \xrightarrow{E} \mathscr{K})$$

— Φ preserves filtered colimits,

$$\Phi(K) = \operatorname{colim}(\mathscr{K}_{fp} \downarrow K \xrightarrow{E} \mathscr{K} \xrightarrow{\Phi} \mathscr{K})$$

Consider now a component of the filtered colimit, $K = \operatorname{colim} a_i$:

$$\begin{array}{lcl} a_0 \xrightarrow{\mathrm{in}_{a_0}} & K \xrightarrow{\quad e \quad} & \Phi(K) \end{array}$$

a_0 is a finitely presentable object and the colimit, $\Phi(K) = \operatorname{colim} \Phi(a_i)$ is filtered, so there is a factorization of the morphism $e \cdot \mathrm{in}_{a_0}$ through a component of the colimit:

$$\begin{array}{ccccc} a_0 & \xrightarrow{m_1} & \Phi(a_1) & & \\ & \searrow^{\mathrm{in}_{a_0}} & & \searrow^{\Phi(\mathrm{in}_{a_1})} & \\ & & K & \xrightarrow{\quad e \quad} & \Phi(K) \end{array}$$

Doing the same for the component in_{a_1} we take:

$$\begin{array}{ccccc} a_0 & \xrightarrow{m_1} & \Phi(a_1) & & \\ & \searrow^{\mathrm{in}_{a_0}} & & \searrow^{\Phi(\mathrm{in}_{a_1})} & \\ & & K & \xrightarrow{\quad e \quad} & \Phi(K) \\ & \nearrow_{\mathrm{in}_{a_1}} & & \nearrow_{\Phi(\mathrm{in}_{a_2})} & \\ a_1 & \xrightarrow[m_2]{} & \Phi(a_2) & & \end{array}$$

If we continue like this, there arises a sequence

$$a_0 \xrightarrow{m_1} \Phi(a_1), \qquad a_1 \xrightarrow{m_2} \Phi(a_2), \qquad a_2 \xrightarrow{m_3} \Phi(a_3), \qquad \cdots \tag{3.4}$$

of morphisms in $\mathscr{K}$, where all the objects a_0, a_1, a_2, ... are finitely presentable.

We call such a sequence of morphisms a *complex*. Using the correspondence between arrows $b \xrightarrow{m} \Phi(a)$ and module elements $a \overset{m}{\rightarrowtail\!\!\!\!\!+} b$, explained in the previous section, we may rephrase the definition in terms of sequences of module elements. Notice that we will eventually need to use the advantages of the latter notation (in particular the possibility of composing arrows with module elements) and the subsequent "calculus of complexes" in order to develop our theory.

Definition 3.1. Given a (flat) module M on a small category $\mathscr{A}$, the category

$$\mathsf{Complex}(M)$$

of *M-complexes and their morphisms* is defined as follows:

(1) Objects, called *M-complexes*, are countable chains of the form

$$\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0$$

A single complex as above will be denoted by $(a_\bullet, m_\bullet)$ for short.

(2) *Morphisms from $(a_\bullet, m_\bullet)$ to $(a'_\bullet, m'_\bullet)$* are sequences $f_n : a_n \longrightarrow a'_n$, denoted by $(f_\bullet)$, such that all squares in the following diagram

$$\begin{array}{ccccccc} \cdots \xrightarrow{m_3} & a_2 & \xrightarrow{m_2} & a_1 & \xrightarrow{m_1} & a_0 \\ & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ \cdots \xrightarrow[m'_3]{} & a'_2 & \xrightarrow[m'_2]{} & a'_1 & \xrightarrow[m'_1]{} & a'_0 \end{array}$$

commute.

For $n \geq 0$, we denote by

$$\mathsf{Complex}_n(M)$$

the category of *n-truncated M-complexes*. Its objects are finite chains

$$a_n \xrightarrow{m_n} a_{n-1} \longrightarrow \cdots \longrightarrow a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0$$

and the morphisms of $\mathsf{Complex}_n(M)$ are defined in the obvious way.

The obvious truncation functors are denoted by

$$\mathrm{pr}_n : \mathsf{Complex}(M) \longrightarrow \mathsf{Complex}_n(M), \quad n \geq 0$$

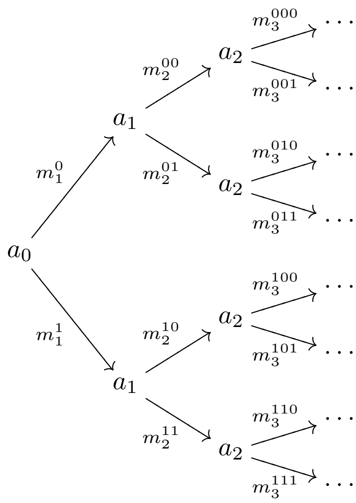
Observe that $\mathsf{Complex}_0(M) = \mathscr{A}$.

Example 3.2. Recall the flat module M of Example 2.11 that corresponds to the finitary endofunctor $X \mapsto X \times X + A$ of sets.

An M-complex

$$\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0$$

can be identified with a “binary tree” of maps of the form



where each path is either infinite or it ends with a generalized element $a_n \longrightarrow A$ of A.

Remark 3.3. Notice that in the alternative notation a morphism of complexes becomes just a sequence of morphisms in $\mathscr{K}$ making the obvious squares commutative:

$$
\begin{array}{ccccccccccc}
a_0 & \xrightarrow{m_1} & \Phi(a_1) & & a_1 & \xrightarrow{m_2} & \Phi(a_2) & & a_2 & \xrightarrow{m_3} & \Phi(a_3) \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \Phi(f_1)} & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \Phi(f_2)} & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \Phi(f_3)} & \cdots \\
a'_0 & \xrightarrow[m'_1]{} & \Phi(a'_1) & & a'_1 & \xrightarrow[m'_2]{} & \Phi(a'_2) & & a'_2 & \xrightarrow[m'_3]{} & \Phi(a'_3)
\end{array}
$$

In this case, stressing the notational distinction, we denote a complex by $a^\bullet$ and the category of complexes by $\mathsf{Complex}(\Phi)$.

Returning to the definition of the final coalgebra, we may now revise our strategy, following the analysis preceding Definition 3.1.

Consider $T \in \mathscr{K}$ to be the colimit of the diagram

$$
\mathsf{Complex}(\Phi) \xrightarrow{\mathrm{pr}_0} \mathscr{K}_{fp} \xrightarrow{E} \mathscr{K}
$$

In order that this definition works, two things would be needed: That this colimit exists in $\mathscr{K}$ (recall that this is not automatically granted when $\mathscr{K}$ is not l.f.p) and that it is the carrier of a coalgebra structure.

Concerning the existence of the above colimit in the finitely accessible $\mathscr{K}$, it would suffice to know that the colimit is filtered, i.e that the indexing category of complexes is filtered. The Strong Solvability Condition on a self-similarity system $(\mathscr{A}, M)$ asserts exactly that.

Definition 3.4. We say that $(\mathscr{A}, M)$ satisfies the *Strong Solvability Condition* if the category $\mathsf{Complex}(M)$ is cofiltered.

Remark 3.5. The Strong Solvability Condition implies that the diagram

$$\Big(\mathsf{Complex}(M)\Big)^{op} \xrightarrow{\mathrm{pr}_0^{op}} \mathscr{A}^{op} \xrightarrow{Y} [\mathscr{A}, \mathsf{Set}]$$

of representables is filtered. Its colimit (a flat functor!) is going to be the carrier of the final coalgebra for $M \otimes -$, see Theorem 3.10 below.

Although the condition is rather strong and hard to verify directly in a general finitely accessible category (and we will seek a weaker one in Section 5), it is easily verified in the realm of locally finitely presentable categories and it can also be employed for studying final coagebras in certain non-l.f.p accessible categories (see Section 4).

Proposition 3.6. Suppose that there exists a coalgebra for $M \otimes -$ and that $\mathscr{A}$ has nonempty finite limits. Then the category $\mathsf{Complex}(M)$ is cofiltered.

Proof. Suppose that $e : X \longrightarrow M \otimes X$ is some coalgebra. The functor X must be flat, hence there exists an element $x_0 \in Xa_0$. Consider the element $e_{a_0}(x_0) \in (M \otimes X)(a_0)$. Since

$$(M \otimes X)(a_0) = \int^a M(a, a_0) \times Xa$$

there exist a_1, $m_1 \in M(a_1, a_0)$ and $x_1 \in Xa_1$ such that the pair (m_1, x_2) represents $e_{a_0}(x_0)$. It is clear that in this way we can construct a complex.

Suppose that

$$D : \mathscr{D} \longrightarrow \mathsf{Complex}(M)$$

with $\mathscr{D}$ finite and nonempty, is given. Let us put

$$\begin{array}{ccc} Dd & & \cdots \xrightarrow{m_3^d} a_2^d \xrightarrow{m_2^d} a_1^d \xrightarrow{m_1^d} a_0^d \\ \Big\downarrow{\scriptstyle D\delta} & = & \qquad\quad \Big\downarrow{\scriptstyle \delta_2} \qquad\quad \Big\downarrow{\scriptstyle \delta_1} \qquad\quad \Big\downarrow{\scriptstyle \delta_0} \\ Dd' & & \cdots \xrightarrow[m_3^{d'}]{} a_2^{d'} \xrightarrow[m_2^{d'}]{} a_1^{d'} \xrightarrow[m_1^{d'}]{} a_0^{d'} \end{array}$$

and observe that, for each $n \geq 0$, its n-th coordinate provides us with a diagram of shape $\mathscr{D}$ in $\mathscr{A}$. Since $\mathscr{A}$ has finite nonempty limits, we can denote, for each $n \geq 0$, by

$$c_n^d : a_n \longrightarrow a_n^d$$

the limit of the n-th coordinate.

For each $n \geq 0$, we define $m_{n+1} \in M(a_{n+1}, a_n)$ as follows: since

$$M(a_{n+1}, a_n) \cong \lim_d M(a_{n+1}, a_n^d)$$

holds by flatness of M, there is a unique m_{n+1} such that the square

$$\begin{array}{ccc} a_{n+1} & \xrightarrow{m_{n+1}} & a_n \\ {\scriptstyle c^d_{n+1}}\downarrow & & \downarrow{\scriptstyle c^d_n} \\ a^d_{n+1} & \xrightarrow[m^d_{n+1}]{} & a^d_n \end{array}$$

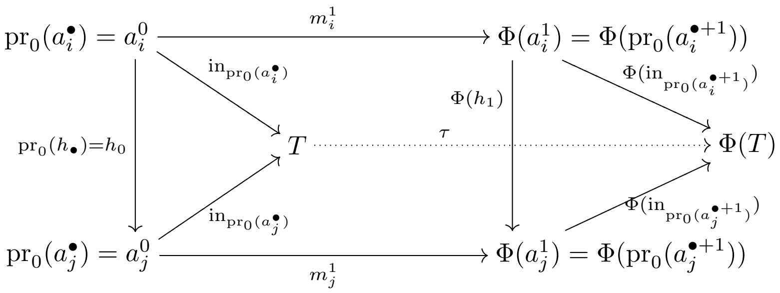
commutes.

The complex $(a_\bullet, m_\bullet)$ defined in the above manner is easily seen to be a limit of $D : \mathscr{D} \longrightarrow \mathsf{Complex}(M)$. This finishes the proof that $\mathsf{Complex}(M)$ is cofiltered, hence $(\mathscr{A}, M)$ satisfies the Strong Sovability Condition. □

Returning now to the issue of defining the coalgebra structure $\tau : T \longrightarrow \Phi(T)$, we will prove that $\Phi(T)$ is a cocone. In this connection recall the proof of the classical (posetal) fixed-point lemma: One shows that $\Phi(x_0)$ is an upper bound for the set defining x_0. Having assumed now that the category $\mathsf{Complex}(\Phi)$ is filtered (and the endofunctor Φ is finitary), we have

$$\Phi(T) = \operatorname{colim} \Phi(\mathrm{pr}_0(a_i^\bullet))$$

Consider the diagram



where, $h_\bullet : a_i^\bullet \longrightarrow a_j^\bullet$ is a diagram in $\mathsf{Complex}(\Phi)$, i.e

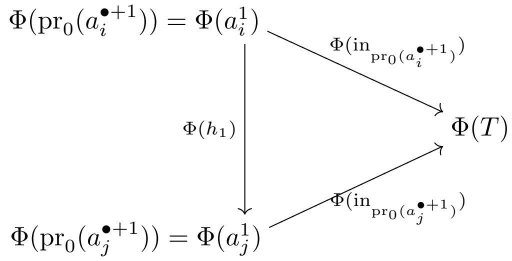
$$\begin{array}{ccccccccc} a_i^0 & \xrightarrow{m_i^1} & \Phi(a_i^1) & , & a_i^1 & \xrightarrow{m_i^2} & \Phi(a_i^2) & , & \cdots \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle \Phi(h_1)} & & {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle \Phi(h_2)} & & \\ a_j^0 & \xrightarrow[m_j^1]{} & \Phi(a_j^1) & , & a_j^1 & \xrightarrow[m_j^2]{} & \Phi(a_j^2) & , & \cdots \end{array} \tag{3.5}$$

such that

$$\mathrm{in}_{\mathrm{pr}_0(a_j^\bullet)} \cdot h_0 = \mathrm{in}_{\mathrm{pr}_0(a_i^\bullet)}$$

If we omit the first square we take a new diagram in $\mathsf{Complex}(\Phi)$. We denote the new complexes with $a_i^{\bullet+1}$ and $a_j^{\bullet+1}$ where, $\partial_0(a_i^{\bullet+1}) = a_i^1$ and $\mathrm{pr}_0(a_j^{\bullet+1}) = a_j^1$.

From the definition of T, we take that the latter complexes participate to the colimit as well, and since Φ preserves filtered colimits we have the following commutative diagram,



Hence,

$$\begin{aligned}\Phi(\mathrm{in}_{\mathrm{pr}_0(a_i^{\bullet+1})}) \cdot m_i^1 &= \Phi(\mathrm{in}_{\mathrm{pr}_0(a_j^{\bullet+1})}) \cdot \Phi(h_1) \cdot m_i^1 \\ &= \Phi(\mathrm{in}_{\mathrm{pr}_0(a_j^{\bullet+1})}) \cdot m_j^1 \cdot h_0\end{aligned}$$

which renders $\Phi(T)$ a cocone.
Finally from the universal property of the colimit T, we take the desired morphism $\tau : T \longrightarrow \Phi(T)$, with

$$\tau \cdot \mathrm{in}_{\mathrm{pr}_0(a_i^{\bullet})} = \mathrm{in}_{\mathrm{pr}_0(a_i^{\bullet+1})} \cdot m_i^1$$

for all i.

Having produced a coalgebra whose carrier is, under suitable assumptions, a filtered colimit of representable functors (equivalently, finitely presentable objects), all that remains is to produce the unique coalgebra homomorphism into it out of an arbitrary coalgebra. To that end we rely on the following fundamental idea of Tom Leinster:

Definition 3.7. The complex $(a_\bullet, m_\bullet)$ together with the sequence (x_n) constructed in the beginning of the proof of 3.6 is called an *e-resolution* of $x_0 \in Xa_0$.

Remark 3.8. The above construction of an e-resolution indicates that a coalgebra $e : X \longrightarrow M \otimes X$ is a system of recursive equations that "varies in time". For at "time" a_0 we can write the system of formal recursive equations

$$\begin{aligned} x_0 &\equiv m_1 \otimes x_1 \\ x_1 &\equiv m_2 \otimes x_2 \\ &\vdots \end{aligned}$$

where (x_n) and $(a_\bullet, m_\bullet)$ form the e-resolution of $x_0 \in Xa_0$. Above, we use the tensor notation to denote, e.g., by $m_1 \otimes x_1$ the element of $\int^a M(a, a_0) \times Xa$ represented by the pair (m_1, x_1).

Of course, any "evolution of time" $f : a_0 \longrightarrow a_0'$ provides us with a compatible corresponding recursive system starting at $x_0' = Xf(x_0) \in Xa_0'$.

Remark 3.9. The proof of the following theorem is a straightforward modification of the proof of Theorem 5.11 of (Leinster (2011)). The reason is that our definition of the carrier of the final coalgebra (as a certain colimit) coincides with the definition of Tom Leinster's (as being pointwise a set of connected components of a certain diagram, see Theorem 2.1 of (Paré (1973)). Observing this, the reasoning of the proof goes exactly as in (Leinster (2011)).

Theorem 3.10. Any $(\mathscr{A}, M)$ satisfying the Strong Solvability Condition admits a final coalgebra for $M \otimes -$.

Proof. Define $T : \mathscr{A} \longrightarrow \mathsf{Set}$ to be the colimit of the diagram

$$\Big(\mathsf{Complex}(M)\Big)^{op} \xrightarrow{\mathrm{pr}_0^{op}} \mathscr{A}^{op} \xrightarrow{Y} [\mathscr{A}, \mathsf{Set}] \tag{3.6}$$

By the Strong Solvability Condition, T is a flat functor, being a filtered colimit of representables. Observe that $x \in Ta$ is an equivalence class of complexes of the form

$$\cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0 = a$$

where two such complexes are equivalent if and only if they both map to a further complex ending at a, via complex morphisms having identity on a as the 0-th component. Thus it is exactly a reduction (due to filteredness) of the description of elements of a final coalgebra that Tom Leinster has for his setting in (Leinster (2011)). We denote equivalence classes by square brackets.

In this representation of the objects of $\mathscr{K}$ as flat functors, the coalgebra structure $\tau : T \longrightarrow M \otimes T$, investigated in the discussion following the proof of Proposition 3.6, has the following objectwise transcription: For each $a \in \mathscr{A}$

$$\tau_a : Ta \longrightarrow (M \otimes T)(a) = \int^{a'} M(a', a) \times Ta'$$

is a map sending the equivalence class

$$[\ \cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0 = a\ ]$$

to the element

$$a_1 \xrightarrow{m_1} a_0\ \otimes [\ \ldots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1\ ]$$

of $(M \otimes T)(a)$ (recall the tensor notation of Remark 3.8).

By Proposition 5.8 of (Leinster (2011)) such τ is a natural isomorphism. That $\tau : T \longrightarrow M \otimes T$ is a final coalgebra follows from Theorem 5.11 of (Leinster (2011)), once we have verified that T is flat. Tom Leinster proves finality with respect to componentwise flat functors so, *a fortiori*, the coalgebra τ is final with respect to coalgebras whose carriers are flat functors. □

Remark 3.11. Observe that (the a-th component of) the mapping $\tau_a : Ta \longrightarrow (M \otimes T)(a)$ is indeed very similar to the coalgebraic structure $\tau = \langle \mathsf{head}, \mathsf{tail} \rangle$ of the final coalgebra of streams of Example 1.1.

In the realm of locally finitely presentable categories, *every* finitary endofunctor admits a final coalgebra. The well-known technique for proving this result is that of 2-categorical limits of locally finitely presentable categories, see, e.g., (Makkai and Paré (1989)) or (Adámek and Rosický (1994)).

Our technique will allow us to give an alternative proof of this theorem, see Corollary 3.12 below. In fact, the colimit of (3.6) gives an explicit description of a final coalgebra.

Corollary 3.12. Every finitary endofunctor of a locally finitely presentable category admits a final coalgebra.

Proof. Recall that the category of the form $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ is locally finitely presentable, if the category $\mathscr{A}$ has *all* finite limits. Denote by $(\mathscr{A}, M)$ the corresponding self-similarity system. We need to show that $\mathsf{Complex}(M)$ is cofiltered.

(1) The category $\mathsf{Flat}(\mathscr{A}, \mathsf{Set}) \simeq \mathsf{Lex}(\mathscr{A}, \mathsf{Set})$ has an initial object, $\perp$, say. Hence the unique morphism $! : \perp \longrightarrow M \otimes \perp$ is a coalgebra and the category $\mathsf{Complex}(M)$ is nonempty.

(2) By Proposition 3.6, the category $\mathsf{Complex}(M)$ has cones for nonempty finite diagrams.

Now use Theorem 3.10. □

Theorem 3.10 provides us with a concrete description of the final coalgebra as the colimit of the filtered diagram

$$\Big(\mathsf{Complex}(M)\Big)^{op} \xrightarrow{\mathrm{pr}_0^{op}} \mathscr{A}^{op} \xrightarrow{Y} [\mathscr{A}, \mathsf{Set}]$$

From that one can easily deduce, for example, the well-known description of the final coalgebra for the endofunctor $X \mapsto X \times X + A$ on Set that we gave in the 3.2.

Remark 3.13. There have been many approaches to giving concrete descriptions of the final coalgebra in various settings. The main vehicle towards such descriptions has been that of the "final sequence", which has been studied exhaustively for the case of finitary endofunctors of sets by J. Worrell ((Worrell (2005))) and generalized to certain l.f.p categories by J. Adámek ((Adámek (2003))). This approach involves the limit L of the ω^{op}-indexed diagram

$$\ldots \longrightarrow \Phi^n(1) \longrightarrow \ldots \longrightarrow \Phi(1) \longrightarrow 1,$$

where 1 is the terminal object of the l.f.p category. This limit may not yield the carrier of the final coalgebra (even in sets, as the example of the "finite powerset" endofunctor shows), but it does so if we keep iterating: Worrell shows that the limit of

$$\ldots \longrightarrow \Phi^n(L) \longrightarrow \ldots \longrightarrow \Phi(L) \longrightarrow L$$

always gives the carrier of the final coalgebra ("the sequence stabilizes after $\omega + \omega$ steps"). Adámek generalizes this to strongly l.f.p categories with a projective generator and finitary endofunctors preserving strong monomorphisms (see (Adámek (2003)) for the involved terminology). In the framework of Adámek's the final coalgebra is a strong subobject of L. Notice here that there is always a canonically defined morphism

$$\sigma : T \longrightarrow L$$

where for each $a \in \mathscr{A}$
$\sigma_a([\ \cdots \xrightarrow{m_3} a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a\ ]) = \langle \dots,\ [\ a_1 \xrightarrow{m_1} a\ , [\ a_2 \xrightarrow{m_2} a_1\ ]]\ ,\ [\ a_1 \xrightarrow{m_1} a\ ]\rangle$
Here the equivalence class

$$[\ a_1 \xrightarrow{m_1} a\ , [\ a_2 \xrightarrow{m_2} a_1\ , ...[\ a_{n+1} \xrightarrow{m_3} a_n\ ]...]]\ ,$$

denotes the element of $\Phi^{n+1}(1)(a) \cong (M \otimes (... \otimes (M \otimes 1)...)(a)$. We can not deduce, exploiting our description, whether this map is injective in any framework more general than that anticipated by Adámek's result.

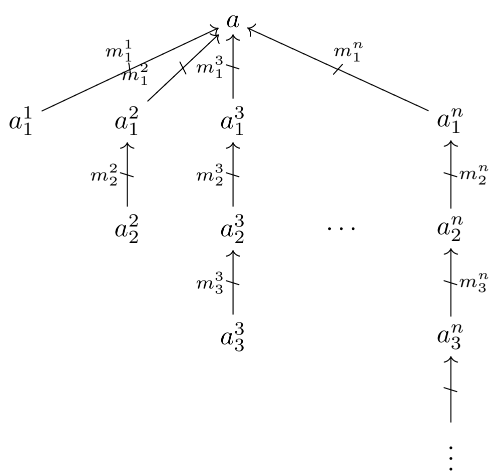
Notice finally that the representation of the finitary endofunctor as a "tensor product" indicates a representation of the elements of L at level a as an infinitely branching, infinitely long tree rooted at a, bearing a close relationship with the elements of the limit L for the finite powerset functor on sets.
Suppose that

$$L = \lim_n M^{(n)} \otimes 1$$

then for each $a \in \mathscr{A}$

$$L(a) = \langle \dots,\ [\ a_1^2 \xrightarrow{m_1^2} a\ , [\ a_2^2 \xrightarrow{m_2^2} a_1^2\ ]]\ ,\ [\ a_1^1 \xrightarrow{m_1^1} a\ ]\rangle$$

such an element, can be represented as:



4. Final coalgebras on posets with distinct endpoints

Before moving to the completely general case of a finitely accessible category and the investigation of more tractable conditions that would ensure the existence of a final coalgebra, let us focus on certain finitary endofunctors on the category of posets with distinct endpoints. This category serves as a suitable ground to test the proposed construction of the final coalgebra, both in terms of verifying the Strong Solvability Condition, as well as for identifying the object we construct with something well-known and classical. In particular, P. Freyd's description of the closed unit interval as a final coalgebra takes place over this category, which falls outside the realm of l.f.p categories.

Example 4.1. Recall the category $\mathsf{Pos}_{0,1}$ of all posets having distinct top and bottom and all monotone maps preserving top and bottom of Example 2.4((6)). Recall that $\mathsf{Pos}_{0,1}$ is finitely accessible but not locally finitely presentable.

Let the functor $\Phi : \mathsf{Pos}_{0,1} \longrightarrow \mathsf{Pos}_{0,1}$ send X to the *smash coproduct*

$$X \vee X$$

of X with itself that is defined as follows: put one copy of X on top of the other one and glue the copies together by identifying top and bottom. More formally, $X \vee X$ is the subposet of $X \times X$ consisting of pairs $(x, 0)$ or $(1, y)$. The pairs $(x, 0)$ are going to be called living in the *left-hand* copy of X and the pairs of the form $(1, y)$ as living in the *right-hand* copy.

Clearly, given a coalgebra $e : X \longrightarrow X \vee X$ and $x \in X$, one can produce at least one infinite sequence

$$x_1 x_2 x_3 \ldots$$

of 0's and 1's as follows: look at $e(x)$ and put $x_1 = 0$ if it is in the left-hand copy of X, put $x_1 = 1$ otherwise. Then regard $e(x)$ as an element of X again, apply e to it to produce x_2, etc.

One needs to show that the binary expansion $e^\dagger(x) = 0.x_1x_2x_3\ldots$ so obtained can be used to define a map $e^\dagger : X \longrightarrow [0, 1]$ in a clash-free way (i.e., regardless of the fact that sometimes we may have a choice in defining $x_k = 0$ or $x_k = 1$). Moreover, the above map $e^\dagger$ is then a witness that the coalgebra

$$t : [0, 1] \longrightarrow [0, 1] \vee [0, 1]$$

where $[0, 1]$ denotes the closed unit interval with the usual order and t given by putting $t(x) = (2x, 0)$ for $0 \leq x \leq 1/2$ and $t(x) = (1, 2x - 1)$ otherwise, is a final coalgebra for Φ.

See (Freyd (1999)), (Freyd (2008)) for more details on the structure of the unit interval arising from such a coalgebraic description.

Proposition 4.2. The self-similarity system $(\mathscr{A}, M)$ corresponding to the functor $\Phi : \mathsf{Pos}_{0,1} \longrightarrow \mathsf{Pos}_{0,1}$ of Example 4.1 satisfies the Strong Solvability Condition.

Proof. Recall that M is defined as

$$M(a, b) = \mathsf{Pos}_{0,1}(b, a \vee a)$$

where the posets a, b are finite (having distinct bottom and top).

A complex $(a_\bullet, m_\bullet)$ is therefore a chain

$$m_1 : a_0 \longrightarrow a_1 \vee a_1, \quad m_2 : a_1 \longrightarrow a_2 \vee a_2, \quad \ldots, \quad m_i : a_i \longrightarrow a_{i+1} \vee a_{i+1}, \quad \ldots$$

of morphisms in $\mathsf{Pos}_{0,1}$.

We have to show that $\mathsf{Complex}(M)$ is cofiltered and we will use the elementary description of complexes of Remark 3.3

(1) $\mathsf{Complex}(M)$ is nonempty.
Let $a_i = 2$, the two-element chain, for every $i \geq 0$ and, for all $i \geq 0$, let $m_i : a_i \longrightarrow a_{i+1} \vee a_{i+1}$ be the unique morphism in $\mathsf{Pos}_{0,1}$. This defines a complex.

(2) $\mathsf{Complex}(M)$ has cones for two-element discrete diagrams.
Suppose $(a_\bullet, m_\bullet)$ and $(a'_\bullet, m'_\bullet)$ are given. Hence we have chains

$$m_1 : a_0 \longrightarrow a_1 \vee a_1, \quad m_2 : a_1 \longrightarrow a_2 \vee a_2, \quad \ldots, \quad m_i : a_i \longrightarrow a_{i+1} \vee a_{i+1}, \quad \ldots$$

and

$$m'_1 : a'_0 \longrightarrow a'_1 \vee a'_1, \quad m'_2 : a'_1 \longrightarrow a'_2 \vee a'_2, \quad \ldots, \quad m'_i : a'_i \longrightarrow a'_{i+1} \vee a'_{i+1}, \quad \ldots$$

Since every pair a_i, a'_i has a cocone in $(\mathsf{Pos}_{0,1})_{fp}$, every pair a_i, a'_i has a coproduct $a_i + a'_i$ in $(\mathsf{Pos}_{0,1})_{fp}$ due to Scott-completeness of $\mathsf{Pos}_{0,1}$, see Example 2.4((6)).
One then uses flatness of M to obtain the desired vertex $(b_\bullet, n_\bullet)$ of a cone as follows: put $b_i = a_i + a'_i$ for all $i \geq 0$ and define $n_i : b_i \longrightarrow b_{i+1} \vee b_{i+1}$ to be the one given by the bijection

$$\mathsf{Pos}_{0,1}(b_i, b_{i+1} \vee b_{i+1}) = \mathsf{Pos}_{0,1}(a_i + a'_i, b_{i+1} \vee b_{i+1}) \cong \mathsf{Pos}_{0,1}(a_i, b_{i+1}) \times \mathsf{Pos}_{0,1}(a'_i, b_{i+1})$$

applied to the obvious pair of morphisms $a_i \longrightarrow b_{i+1}$, $a'_i \longrightarrow b_{i+1}$.

(3) $\mathsf{Complex}(M)$ has cones for parallel pairs.
This follows immediately from the following claim:

There are no serially commutative squares

$$\begin{array}{ccc} X & \overset{u}{\underset{d}{\rightrightarrows}} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle r} \\ Z \vee Z & \overset{h \vee h}{\underset{l \vee l}{\rightrightarrows}} & W \vee W \end{array} \tag{4.7}$$

whenever the maps u, d cannot be coequalized.

Notice first that both $h \vee h$ and $l \vee l$ map the "middle element" $(1, 0)$ of $Z \vee Z$ to the respective "middle element" in $W \vee W$.
Next notice that the only reason for which u and d cannot be coequalized is that some $x \in X$ is sent to 0 by d and to 1 by u. Fix this x, and notice that equations $ru(x) = 1$ and $rd(x) = 0$ hold.
Notice also that

$$H_h = \{z \in Z \vee Z \mid (h \vee h)(z) = 1\}$$

is a proper subset of $\{z \in Z \vee Z \mid z \geq m\}$ where m denotes the "middle element" of $Z \vee Z$.
Similarly,

$$H_l = \{z \in Z \vee Z \mid (l \vee l)(z) = 0\}$$

is a proper subset of $\{z \in Z \vee Z \mid z \leq m\}$.

In particular, $H_h \cap H_l = \emptyset$.

Suppose that the diagram (4.7) serially commutes. Then $s(x) \in H_h \cap H_l$, a contradiction.

□

We indicate how the description of the final coalgebra for the squaring functor on the category $\mathsf{Pos}_{0,1}$ that we gave in Example 4.1 corresponds to the description given by the proof of Theorem 3.10.

We denote the module, corresponding to the squaring functor $X \mapsto X \vee X$, by M. Observe that

$$M(a,b) = \mathsf{Pos}_{0,1}(b, a \vee a)$$

holds.

Recall that by Remark 2.6 there is an equivalence

$$\mathsf{Pos}_{0,1} \simeq \mathsf{Flat}((\mathsf{Pos}_{0,1})^{op}_{fp}, \mathsf{Set})$$

of categories that we will use now: the flat functor $I : (\mathsf{Pos}_{0,1})^{op}_{fp} \longrightarrow \mathsf{Set}$ that is the carrier of the final coalgebra for $M \otimes -$ is transferred by the above equivalence to the poset

$$I \star E$$

see Remark 2.6. We define now the map

$$\mathsf{beh} : I \star E \longrightarrow [0,1]$$

where $[0,1]$ is the unit interval with the coalgebra structure described in Example 4.1.

The mapping beh assigns to the equivalence class

$$\Big[[(a_\bullet, m_\bullet)], x \in a_0\Big] \in I \star E$$

a dyadic expansion that encodes the behaviour of $x \in a_0$ as follows: we know that a complex $(a_\bullet, m_\bullet)$ is a chain

$$m_1 : a_0 \longrightarrow a_1 \vee a_1, \quad m_2 : a_1 \longrightarrow a_2 \vee a_2, \quad \ldots, \quad m_i : a_i \longrightarrow a_{i+1} \vee a_{i+1}, \quad \ldots$$

of morphisms in $\mathsf{Pos}_{0,1}$. The morphism m_1 sends x to the left-hand copy or to the right-hand copy of a_1, so it gives rise to a binary digit $k_1 \in \{0,1\}$ and a new element $x_1 \in a_1$. (If $m_1(x)$ is in the glueing of the two copies of a_1, choose 0 or 1 arbitrarily). Iterating gives a binary representation $0.k_1k_2\ldots$ of an element of $[0,1]$.

Proposition 4.3. The map

$$\mathsf{beh} : I \star E \longrightarrow [0,1]$$

is well-defined and a bijection.

Proof.

(1) beh is well-defined: Let $\Big[[(a_\bullet, m_\bullet)], x \in a_0\Big] = \Big[[(a'_\bullet, m'_\bullet)], x' \in a'_0\Big]$, then there is an element $\Big[[(c_\bullet, q_\bullet)], y \in c_0\Big]$ of the colimit and a zig-zag:

$$\begin{array}{ccccccc} a_0 & \xrightarrow{m_1} & a_1 \vee a_1 & \quad & a_1 & \xrightarrow{m_2} & a_2 \vee a_2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ c_0 & \xrightarrow{q_1} & c_1 \vee c_1 & & c_1 & \xrightarrow{q_2} & c_2 \vee c_2 \quad \cdots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ a'_0 & \xrightarrow{m'_1} & a'_1 \vee a'_1 & & a'_1 & \xrightarrow{m'_2} & a'_2 \vee a'_2 \end{array}$$

such that all the squares are commutative.
Observe that, in order to have the commutativity of the above squares, the morphisms m_i, q_i, m'_i, $i = 1, 2, \ldots$ must have the same "behaviour". This means that if, e.g., the morphism m_1 sends x to the left-hand copy of $a_1 \vee a_1$ then also the q_1, m'_1 will send the corresponding elements to the left-hand copy of $c_1 \vee c_1$ and $a'_1 \vee a'_1$ respectively. So, we take the same binary representation in $[0, 1]$, i.e., the equality

$$\mathsf{beh}([(a_\bullet, m_\bullet)], x \in a_0) = \mathsf{beh}([(a'_\bullet, m'_\bullet)], x' \in b_0)$$

holds.

(2) beh is one to one:
The key-point here is that there is a morphism $f : 5 \longrightarrow 5 \vee 5$, where 5 is the linear order with five elements, such that for each $m_i : a_i \longrightarrow a_{i+1} \vee a_{i+1}$ there is a commutative square

$$\begin{array}{ccc} a_i & \xrightarrow{m_i} & a_{i+1} \vee a_{i+1} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h' \vee h'} \\ 5 & \xrightarrow[f]{} & 5 \vee 5 \end{array}$$

Suppose that $\{0, t_1, t_2, t_3, 1\}$ are the elements of 5, then the elements of $5 \vee 5$ will be denoted by $\{0, t_1^L, t_2^L, t_3^L, c', t_1^R, t_2^R, t_3^R, 1\}$.
We define:

$$f(t) = \begin{cases} 0, & \text{if } t = 0 \\ 1, & \text{if } t = 1 \\ t_2^L, & \text{if } t = t_1 \\ t_2^R, & \text{if } t = t_3 \\ c', & \text{if } t = t_2 \end{cases} \qquad h(x) = \begin{cases} 0, & \text{if } m_i(x) = 0 \\ 1, & \text{if } m_i(x) = 1 \\ t_1, & \text{if } m_i(x) \in a_{i+1}^L \\ t_3, & \text{if } m_i(x) \in a_{i+1}^R \\ t_2, & \text{if } m_i(x) = c \end{cases}$$

$$h'(z) = \begin{cases} 0, & \text{if } z = 0 \\ 1, & \text{if } z = 1 \\ t_2, & \text{otherwise} \end{cases}$$

where L, R denotes the left-hand and the right-hand copy and c, c' are the glueing points of $a_{i+1} \vee a_{i+1}$ and $5 \vee 5$, respectively. From the above it is easy to verify the commutativity of the square.
Now, if $\mathsf{beh}([(a_\bullet, m_\bullet)], x \in a_0) = \mathsf{beh}([(b_\bullet, n_\bullet)], y \in b_0)$, i.e., if the binary representations are the same, we can choose without loss of generality the m_i and n_i to send the x_i, y_i to the same copy left-hand or right-hand, respectively. (Hence we avoid the

case one of them sending an element to the glueing point). Using commutativity of the above square we have that all the following squares commute:

$$
\begin{array}{ccccccc}
a_0 & \xrightarrow{m_1} & a_1 \vee a_1 & \qquad & a_1 & \xrightarrow{m_2} & a_2 \vee a_2 \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h' \vee h'} & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h' \vee h'} \\
5 & \xrightarrow{f} & 5 \vee 5 & & 5 & \xrightarrow{f} & 5 \vee 5 & \quad \cdots \\
{\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h' \vee h'} & & {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h' \vee h'} \\
b_0 & \xrightarrow{n_1} & b_1 \vee b_1 & & b_1 & \xrightarrow{n_2} & b_2 \vee b_2
\end{array}
$$

From this we deduce that there is a zig-zag between the two complexes,$(a_\bullet, m_\bullet)$,$(b_\bullet, n_\bullet)$. Therefore, the equality

$$[(a_\bullet, m_\bullet)] = [(b_\bullet, n_\bullet)]$$

holds.

(3) beh is epi: For each binary representation $0.k_1k_2\ldots$ of an element of $[0,1]$ we can find an element of the colimit, using the three-element linear order 3, and a sequence

$$m_1 : 3 \longrightarrow 3 \vee 3, \quad m_2 : 3 \longrightarrow 3 \vee 3, \quad \ldots \quad m_i : 3 \longrightarrow 3 \vee 3, \quad \ldots$$

of morphisms, where each m_i assigns the middle element of 3, to the middle element in the left-hand copy of $3 \vee 3$ if $k_i = 0$, or the middle element in the right-hand copy if $k_i = 1$.

□

We continue showing that our approach ensures the existence of the final coalgebra in a number of other situations that are not covered by Corrolary 3.12, nor by guessing what the final coalgebra is and then verifying its universal property. First notice that the category $\mathsf{Pos}_{0,1}$ has products and coproducts, thus we may define polynomial endofunctors on it, of the form

$$X \mapsto A_n \times X^n + ... + A_1 \times X + A_0$$

(and such endofunctors are finitary). Notice also that the coproduct is formed by two disjoint copies glued together at the top and bottom.

Proposition 4.4. Every polynomial endofunctor Φ on posets with distinct endpoints admits a final coalgebra.

Proof. One may easily see that: The category of coalgebras is non-empty, thus so is the category of complexes and the existence of coproducts ensures that discrete diagrams of complexes have cocones.

Moreover, as it was the case with Freyd's endofunctor, there are no parallel morphisms of complexes, whenever parallel pairs of maps between the vertices of the complexes can not be coequalized: If in the diagram

$$\begin{array}{ccc} a & \overset{u}{\underset{v}{\rightrightarrows}} & b \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle r} \\ A_n \times c^n + ... + A_1 \times c + A_0 & \overset{\Phi(h)}{\underset{\Phi(l)}{\rightrightarrows}} & A_n \times d^n + ... + A_1 \times d + A_0 \end{array} \tag{4.8}$$

an element of a is sent to 0 by u and to 1 by v, then the module element s has to send this element

— either to 0, hence it remains 0 in $\Phi(d)$, or
— to 1 and, again, it remains 1 in $\Phi(d)$
— or to a "middle" element, which then is acted upon by the images of h and l under the polynomial endofunctor. The middle element can be either in the "constant term" A_0 of the polynomial, which is then mapped to itself by both maps (hence can not be both 0 and 1), or is some $(a_n, x, x, ..., x) \in A_n \times X^n$, which is then mapped to $(a_n, \Phi(h)(x), ..., \Phi(h)(x))$ by the one map and to $(a_n, \Phi(l)(x), ..., \Phi(l)(x))$ by the other map. Hence we can not have one of them be 0 and the other one be 1, in order to have serial commutativity, because that would require a_n to be 0 for the one case and 1 for the other.

Thus the polynomial endofunctor satisfies Strong Solvability Condition, hence it admits a final coalgebra. □

Remark 4.5. It seems possible that the above proposition applies to other categories of structures (of the kind that together with their homomorphisms form an l.f.p category, see the Introduction), where we specify that two constants are distinct and, moreover, have some reasonable behaviour of coproducts. In particular it applies to "bipointed sets".

A further situation where our approach could be put at work is the following: Let Φ be an arbitrary finitary endofunctor on $\mathsf{Pos}_{0,1}$ (or on a category as those discussed in the previous remark). Since products exists in this category, we may ask whether the final coalgebra for the endofunctor $C \times \Phi(-)$ exists. It is well-known though that this question is tantamount to asking whether the cofree coalgebra on the object C exists. Indeed, a final coalgebra $\tau_C: \hat{C} \longrightarrow C \times \Phi(\hat{C})$ amounts to a pair of morphisms $\epsilon_C: \hat{C} \longrightarrow C$ and $\kappa: \hat{C} \longrightarrow \Phi(\hat{C})$, so that for every coalgebra $X \longrightarrow C \times \Phi(X)$ (i.e, pair of morphisms $f: X \longrightarrow C$ and coalgebra structure $\rho_X: X \longrightarrow \Phi X$, there is a unique coalgebra morphism

$$\begin{array}{ccc} X & \xrightarrow{(f,\rho_x)} & C \times \Phi(X) \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times\Phi(\hat{f})} \\ \hat{C} & \xrightarrow[(\epsilon_C,\kappa)]{} & C \times \Phi(\hat{C}) \end{array} \tag{4.9}$$

In other words, for each C, there is a Φ-coalgebra $\hat{C}, \kappa$ and a map ϵ_C (the counit) so that for every $f: X \longrightarrow C$ there is a unique coalgebra map $\hat{f}: (X, \rho_X) \longrightarrow (\hat{C}, \kappa)$, such that

$f = \epsilon_C \cdot f$. A detailed study of the construction of cofree coalgebras in the framework of accessible categories will be presented elsewhere (see (Karazeris, Matzaris and Velebil)). Nevertheless, with the methods developed here we can conclude the following:

Proposition 4.6. For every finitary endofunctor Φ on $\mathsf{Pos}_{0,1}$ and object C, the functor $C \times \Phi(-)$ admits a final coalgebra. In particular the cofree Φ-coalgebra over C exists.

Proof. As in the proof of the previous proposition, if a pair of parallel morphisms between a and b can not be coequalized (because for some $x \in a$ we have $u(x) = 0$ and $v(x) = 1$) then there is no serially commutative

$$\begin{array}{ccc} a & \overset{u}{\underset{v}{\rightrightarrows}} & b \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle r} \\ C \times \Phi a_1 & \overset{\mathrm{id}\times\Phi(h)}{\underset{\mathrm{id}\times\Phi(l)}{\rightrightarrows}} & C \times b_1 \end{array} \tag{4.10}$$

for that would that we have both $s(x) = 0$ and $s(x) = 1$. Hence the Strong Solvability Condition applies. □

5. The Weak Solvability Condition

Cofilteredness of the category $\mathsf{Complex}(M)$ may be hard to verify in the absence of finite limits in $\mathscr{A}$. We give here a weaker condition that is easier to verify. In particular, we are going to replace the Strong Solvability Condition by a condition of the same type but "holding just on the head of complexes". This whole section is devoted to finding conditions of "how to propagate from the head of a complex to the whole complex". Proving the existence of a final coalgebra will require though some extra finiteness condition on the module M, see Definition 5.9. Our condition is a weakening of that considered by Tom Leinster (Leinster (2011)). The main result of this section, Theorem 5.15, then shows that this finiteness condition allows us to conclude that a final coalgebra exists. Our argument applies to self-similarity systems considered by Tom Leinster (Leinster (2011)) and therefore strenghtens his result on the existence of final coalgebras for self-similarity systems.

The key tool for the propagation technique is "König's Lemma for preorders", see Theorem 5.6 below. The result relies on a topological fact proved by Arthur Stone in (Stone (1979)).

To be able to state the weak condition we first need to generalize filteredness of a category to filteredness of a functor.

Definition 5.1. A functor $F : \mathscr{X} \longrightarrow \mathscr{Y}$ is called *filtering*, if there exists a cocone for the composite $F \cdot D$, for every functor $D : \mathscr{D} \longrightarrow \mathscr{X}$ with $\mathscr{D}$ finite.

A functor F is called *cofiltering* if F^{op} is filtering.

Remark 5.2. Hence a category $\mathscr{X}$ is filtered if and only if the identity functor Id : $\mathscr{X} \longrightarrow \mathscr{X}$ is filtering.

A natural candidate for a weaker form of solvability condition is the following one.

Definition 5.3. We say that $(\mathscr{A}, M)$ satisfies the *Weak Solvability Condition* if the functor

$$\mathrm{pr}_0 : \mathsf{Complex}(M) \longrightarrow \mathsf{Complex}_0(M)$$

is cofiltering.

In particular, observe that the Weak Solvability Condition holds when the category $\mathscr{A}$ is cofiltered.

Remark 5.4. In elementary terms, the Weak Solvability Condition says the following three conditions:

(1) The category $\mathscr{A}$ is non-empty.
(2) For every pair $(a_\bullet, m_\bullet)$, $(a'_\bullet, m'_\bullet)$ in $\mathsf{Complex}(M)$ there is a span

$$\begin{array}{ccc} & & a_0 \\ & \overset{f}{\nearrow} & \\ b & & \\ & \underset{f'}{\searrow} & \\ & & a'_0 \end{array}$$

in $\mathscr{A}$.
(3) For every parallel pair of the form

$$(a_\bullet, m_\bullet) \underset{(v_\bullet)}{\overset{(u_\bullet)}{\rightrightarrows}} (a'_\bullet, m'_\bullet)$$

in $\mathsf{Complex}(M)$ there is a fork

$$b \xrightarrow{\ f\ } a_0 \underset{v_0}{\overset{u_0}{\rightrightarrows}} a'_0$$

in $\mathscr{A}$.

Observe that, since we assume that $\mathsf{Complex}(M)$ is nonempty i.e that at least one object of $\mathscr{K}$ carries a coalgebra structure, the above condition (1) is satisfied: the category $\mathscr{A}$ is nonempty.

Observe that if $(\mathscr{A}, M)$ satisfies the Strong Solvability Condition, it does satisfy the Weak Solvability Condition. In fact, in this case *every* functor $\mathrm{pr}_n : \mathsf{Complex}(M) \longrightarrow \mathsf{Complex}_n(M)$ is cofiltering. The following result shows that the Weak Solvability Condition can be formulated in this way.

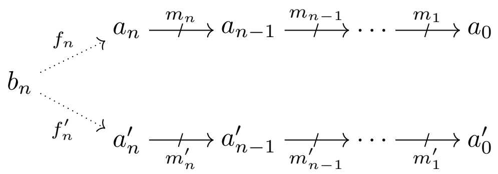
Proposition 5.5. The following are equivalent:

(1) The Weak Solvability Condition.
(2) The functors $\mathrm{pr}_n : \mathsf{Complex}(M) \longrightarrow \mathsf{Complex}_n(M)$ are cofiltering for all $n \geq 0$.

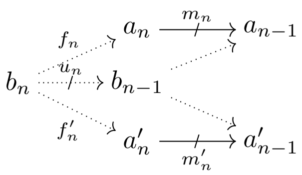
Proof. That (2) implies (1) is clear. To prove the converse, we need to verify the following three properties:

(a) Every category $\mathsf{Complex}_n(M)$ is non-empty. This is clear: we assume that $\mathsf{Complex}(M)$ is non-empty.

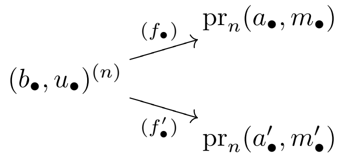
(b) Every pair $\mathrm{pr}_n(a_\bullet, m_\bullet)$, $\mathrm{pr}_n(a'_\bullet, m'_\bullet)$ in $\mathsf{Complex}_n(M)$ has a cone.
Observe that, due to Weak Solvability Condition applied at stage n, we have the following diagram



Since the functor $M(b_n, -)$ is flat, the pair $m_n@f_n \in M(b_n, a_{n-1})$, $m'_n@f'_n \in M(b_n, a'_{n-1})$ of its elements has a cone:



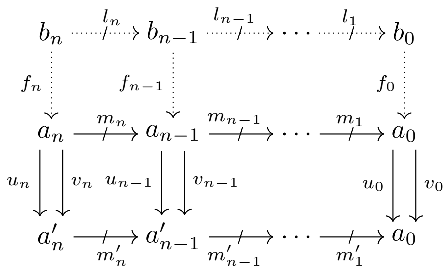
If we proceed like this down to zero we obtain the desired vertex $(b_\bullet, u_\bullet)^{(n)}$ in $\mathsf{Complex}_n(M)$:



(c) For every parallel pair of the form

$$\mathrm{pr}_n(a_\bullet, m_\bullet) \underset{\mathrm{pr}_n(v_\bullet)}{\overset{\mathrm{pr}_n(u_\bullet)}{\rightrightarrows}} \mathrm{pr}_n(a'_\bullet, m'_\bullet)$$

in $\mathsf{Complex}_n(M)$, there is a fork.
Consider the following diagram:



Again, start at stage n, use the Weak Solvability Condition there to obtain f_n, and then use flatness of $M(b_n, -)$ to obtain l_n and f_{n-1}. Proceed like this down to zero and obtain the desired fork

$$(b_\bullet, l_\bullet)^{(n)} \overset{(f_\bullet)^{(n)}}{\dashrightarrow} \mathrm{pr}_n(a_\bullet, m_\bullet) \underset{\mathrm{pr}_n(v_\bullet)}{\overset{\mathrm{pr}_n(u_\bullet)}{\rightrightarrows}} \mathrm{pr}_n(a'_\bullet, m'_\bullet)$$

in $\mathsf{Complex}_n(M)$.

This finishes the proof. □

In the proof that the Weak Solvability Condition implies the Strong one, we will need to use "König's Lemma" for preorders that we formulate in Theorem 5.6 below.

Recall that a *preorder* $\langle X, \sqsubseteq \rangle$ is a set X equipped with a reflexive, transitive binary relation $\sqsubseteq$.

Recall also that a subset $B \subseteq X$ of a preorder is called *downward-closed*, if for every $b \in B$ and $b' \sqsubseteq b$ we have $b' \in B$. The dual notion is called *upward-closed*.

A subset S of a preorder $\langle X, \sqsubseteq \rangle$ is called *final* if for every $x \in X$ there exists $s \in S$ with $x \sqsubseteq s$.

Theorem 5.6. Suppose that

$$\cdots \longrightarrow \mathscr{P}_{n+1} \xrightarrow{p_n^{n+1}} \mathscr{P}_n \xrightarrow{p_{n-1}^{n}} \cdots \xrightarrow{p_0^1} \mathscr{P}_0 \tag{5.11}$$

is a chain of preorders and monotone maps, that satisfies the following two conditions:

(1) Every $\mathscr{P}_n$ has a nonempty finite final subset.

(2) The image of any upward-closed set under $p_n^{n+1} : \mathscr{P}_{n+1} \longrightarrow \mathscr{P}_n$ is upward-closed.

Then the limit $\lim \mathscr{P}_n$ is nonempty, i.e., there is a sequence (x_n) with $p_n^{n+1}(x_{n+1}) = x_n$ holding for every $n \geq 0$.

The proof of Theorem 5.6 will rely on some facts from General Topology that we recall now. As a reference to topology we refer to the book (Engelking (1989)).

Recall that every preorder $\langle X, \sqsubseteq \rangle$ can be equipped with the *lower topology* $\tau_\sqsubseteq$, if we declare the open sets to be exactly the downward closed sets.

Observe that a set B is closed in the topology $\tau_\sqsubseteq$ if and only if it is upward-closed.

Proof of Theorem 5.6. The assumptions (1) and (2) of the statement of the theorem assure that each $\mathscr{P}_n$ is a nonempty compact space in its lower topology and each p_n^{n+1} is a closed continuous map (i.e., on top of continuity, the image of a closed set is a closed set). By result of Arthur Stone (Stone (1979)), Theorem 2, any ω^{op}-chain of nonempty compact spaces and closed continuous maps has a nonempty limit. Therefore $\lim \mathscr{P}_n$ is nonempty.

Remark 5.7. Of course, Theorem 5.6 holds whenever Conditions (1) and (2) hold "cofinally", i.e., whenever there exists n_0 such that Conditions (1) and (2) hold for all $n \geq n_0$.

Notation 5.8. For any diagram $D : \mathscr{D} \longrightarrow \mathsf{Complex}(M)$ with $\mathscr{D}$ finite, let $\mathscr{P}_n^D$ denote the following preorder:

(1) Points of $\mathscr{P}_n^D$ are cones for the composite $\mathrm{pr}_n \cdot D : \mathscr{D} \longrightarrow \mathsf{Complex}_n(M)$.

(2) The relation $c \sqsubseteq_n c'$ holds in $\mathscr{P}_n^D$ if and only if the cone c factors through the cone c'.

For each $n \geq 0$ denote by

$$p_n^{n+1} : \mathscr{P}_{n+1}^D \longrightarrow \mathscr{P}_n^D$$

the obvious restriction map and observe that it is monotone.

Also observe that the Weak Solvability Condition guarantees that every preorder $\mathscr{P}_n^D$ is nonempty by Proposition 5.5. The Weak Solvability Condition alone does not imply the Strong one — the self-similarity system $(\mathscr{A}, M)$ has to fulfill additional conditions that will allow us to apply Theorem 5.6.

Definition 5.9. We say that the module M is *compact*, if the preorder $\mathscr{P}_n^D$ has a nonempty finite final subset, for each $n \geq 0$ and each finite nonempty diagram $D : \mathscr{D} \longrightarrow \mathsf{Complex}(M)$.

First we give easy examples of compact modules.

Example 5.10.

(1) Every module M on a finitely complete category $\mathscr{A}$ is compact: in fact, in this case every preorder $\mathscr{P}_n^D$ has a one-element final set.
(2) If the module M is finite in the sense of (Leinster (2011)), i.e., if every functor $M(-, b) : \mathscr{A}^{op} \longrightarrow \mathsf{Set}$ has a finite category of elements, then it is compact.

Nontrivial examples of compact modules will follow later from Proposition 5.12, see Example 5.14. We need to recall the concept of a factorization system for cocones first. For details, see, e.g., Chapter IV of (Adámek, Herrlich and Strecker (1990)).

Definition 5.11. Let $\mathscr{K}$ be a finitely accessible category.

(1) We say that a cocone $c_d : Dd \longrightarrow X$ is *jointly epi* if, for every parallel pair u, v, the equality $u \cdot c_d = v \cdot c_d$ for all d implies that $u = v$ holds.
(2) We say that $\mathscr{K}$ is a *(finite jointly epi, extremal mono)-category* if the following two conditions are satisfied:
   (a) Every cocone $c_d : Dd \longrightarrow X$ for a finite diagram can be factored as

   $$Dd \xrightarrow{e_d} Z \xrightarrow{j} X$$

   where e_d is jointly epi and j is extremal mono.
   (b) For every commutative square

   $$\begin{array}{ccc} Dd & \xrightarrow{e_d} & X \\ {\scriptstyle f_d}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[j]{} & B \end{array} \qquad \text{(for all } d\text{)}$$

   where e_d is a jointly epi cocone and j is extremal mono, there is a unique diagonal $m : X \longrightarrow A$ making the obvious triangles commutative.

(3) We say that $\mathscr{K}_{fp}$ is *finitely cowellpowered*, if every finite diagram $D : \mathscr{D} \longrightarrow \mathscr{K}_{fp}$ admits (up to isomorphism) only a nonempty finite set of jointly epi cocones.

Proposition 5.12. Suppose the finitely accessible category $\mathscr{K}$ satisfies the following conditions:

(1) $\mathscr{K}$ is a (finite jointly epi, extremal mono)-category.
(2) $\mathscr{K}_{fp}$ is finitely cowellpowered.

Suppose that a finitary functor $\Phi : \mathscr{K} \longrightarrow \mathscr{K}$ preserves extremal monos coming out of finitely presentable objects. Then the flat module corresponding to Φ is compact.

Proof. We will use the description of complexes from Remark 3.3.

Let $D : \mathscr{D} \longrightarrow \mathsf{Complex}(M)$ be a finite nonempty diagram. Choose any $n \geq 0$ and denote the value of the composite $\mathrm{pr}_n \cdot D$ by commutative squares

$$
\begin{array}{c}
\mathrm{pr}_n \cdot Dd \\
\Big\downarrow {\scriptstyle \mathrm{pr}_n \cdot D\delta} \\
\mathrm{pr}_n \cdot Dd'
\end{array}
\quad = \quad
\begin{array}{ccc}
a_0^d & \xrightarrow{m_1^d} & \Phi(a_1^d) \\
{\scriptstyle \delta_0}\uparrow & & \uparrow{\scriptstyle \Phi(\delta_1)} \\
a_0^{d'} & \xrightarrow[m_0^{d'}]{} & \Phi(a_1^{d'})
\end{array}
\quad
\begin{array}{ccc}
a_1^d & \xrightarrow{m_2^d} & \Phi(a_2^d) \\
{\scriptstyle \delta_1}\uparrow & & \uparrow{\scriptstyle \Phi(\delta_2)} \\
a_1^{d'} & \xrightarrow[m_2^{d'}]{} & \Phi(a_2^{d'})
\end{array}
\quad \cdots \quad
\begin{array}{ccc}
a_{n-1}^d & \xrightarrow{m_n^d} & \Phi(a_n^d) \\
{\scriptstyle \delta_{n-1}}\uparrow & & \uparrow{\scriptstyle \Phi(\delta_n)} \\
a_{n-1}^{d'} & \xrightarrow[m_n^{d'}]{} & \Phi(a_n^{d'})
\end{array}
$$

in $\mathscr{K}$. We will construct the finite nonempty *initial* (notice the change of the variance: $\mathscr{A}$ is $\mathscr{K}_{fp}^{op}$) family of cocones for $\mathrm{pr}_n \cdot D$ by proceeding from $i = n - 1$ downwards to 0 as follows:

For every jointly epi cocone $e_{i+1} : a_{i+1}^d \longrightarrow z_{i+1}$ choose all jointly epi cocones $e_i^d : a_i^d \longrightarrow z_i$ and all connecting morphisms $c_{i+1} : z_i \longrightarrow \Phi(z_{i+1})$ making the following diagram

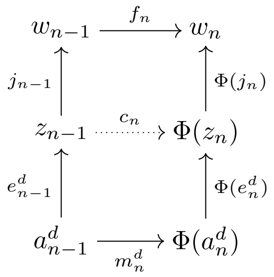
$$
\begin{array}{ccc}
z_i & \overset{c_{i+1}}{\dashrightarrow} & \Phi(z_{i+1}) \\
{\scriptstyle e_i^d}\,\vdots\uparrow & & \uparrow{\scriptstyle \Phi(e_{i+1}^d)} \\
a_i^d & \xrightarrow[m_{i+1}^d]{} & \Phi(a_{i+1}^d)
\end{array}
$$

commutative. Observe that there is at least one such pair: the factorization of the cocone $\Phi(e_{i+1}^d) \cdot m_{i+1}^d$ into a jointly epi and extremal mono. Since every cocone e_i^d is jointly epi, the corresponding c_{i+1} is determined uniquely.

We claim that the above nonempty finite family of cocones for $\mathrm{pr}_n \cdot D$ is initial. To that end, consider any cocone

$$
\begin{array}{ccc}
w_0 & \xrightarrow{f_1} & \Phi(w_1) \\
{\scriptstyle g_0^d}\uparrow & & \uparrow{\scriptstyle \Phi(g_1^d)} \\
a_0^d & \xrightarrow[m_0^d]{} & \Phi(a_1^d)
\end{array}
\quad
\begin{array}{ccc}
w_1 & \xrightarrow{f_2} & \Phi(w_2) \\
{\scriptstyle g_1^d}\uparrow & & \uparrow{\scriptstyle \Phi(g_2^d)} \\
a_1^d & \xrightarrow[m_2^d]{} & \Phi(a_2^d)
\end{array}
\quad \cdots \quad
\begin{array}{ccc}
w_{n-1} & \xrightarrow{f_n} & \Phi(w_n) \\
{\scriptstyle g_{n-1}^d}\uparrow & & \uparrow{\scriptstyle \Phi(g_n^d)} \\
a_{n-1}^d & \xrightarrow[m_n^d]{} & \Phi(a_n^d)
\end{array}
$$

for $\mathrm{pr}_n \cdot D$. Factorize the cocone g_n^d into a jointly epi $e_n^d : a_n^d \longrightarrow z_n$ followed by an extremal mono $j_n : z_n \longrightarrow w_n$. Do the same thing for the cocone g^d_{n-1} and then use the diagonalization property to obtain the desired $c_n : z_{n-1} \longrightarrow \Phi(z_n)$



using the fact that $\Phi(j_n)$ is extremal mono by assumption. Proceed like this downwards to 0 and obtain thus one of the above chosen cocones through which the given cocone of g factorizes. □

Corollary 5.13. Every flat module on Lin is compact.

Proof. We indicate that the category Lin of all linear orders and all monotone maps fulfills the assumptions of the above proposition.

(1) Jointly epi cocones $e_d : Dd \longrightarrow X$ are exactly those where (the underlying set of) X is the union of the images of all Dd.
(2) A monotone map $j : A \longrightarrow B$ is an extremal mono if and only if j is injective and the linear order on A is that induced by B.
(3) An extremal mono coming out of a finite linear order is split: Consider $j : A \longrightarrow B$, an extremal mono with $A = \{x_1, ..., x_n\}$ finite. Then define $s : B \longrightarrow A$ with $s(y) = x_i$, when i is the smallest index such that $y \leq j(x_i)$. Then obviously s is an order-preserving splitting of j.

From the above it is clear that Lin is a (finite jointly epi, extremal mono)-category and that Lin_{fp} is finitely cowellpowered. Moreover, extremal monos coming out of finite linear orders are preserved by any functor. □

Example 5.14. To give various examples of finitary functors, we need to introduce the following notation: given linear orders X and Y we denote by

$$X \,;\, Y \quad \text{(read: } X \textit{ then } Y\text{)}$$

the linear order on the disjoint union of (the underlying sets of) X and Y by putting every element of X to be lower than any element of Y and leaving the linear orders of X and Y unchanged.

The second construction is that of *ordinal product*, by

$$X * Y$$

we denote the linear order on the cartesian product of (underlying sets of) X and Y where we replace each element of Y by a disjoint copy of X. More precisely, $(x, y) < (x', y')$ holds if and only if either $x < x'$ holds or $x = x'$ and $y < y'$.

It can be proved easily that, for example, the following two assignments

$$X \mapsto X * \omega, \quad X \mapsto (X * \omega)\ ;\ 1$$

where ω is the first countable ordinal and 1 denotes the one-element linear order, are finitary functors and they both preserve extremal monos.

Our main result on compact modules is the following one.

Theorem 5.15. Suppose that M is a compact module. Then the Weak Solvability Condition implies the Strong one.

Proof. We know that $\mathsf{Complex}(M)$ is nonempty. We have to construct a cone for every diagram $D : \mathscr{D} \longrightarrow \mathsf{Complex}(M)$ with $\mathscr{D}$ finite nonempty.

Form the corresponding chain

$$\cdots \longrightarrow \mathscr{P}^D_{n+1} \xrightarrow{p^{n+1}_n} \mathscr{P}^D_n \xrightarrow{p^n_{n-1}} \cdots \xrightarrow{p^1_0} \mathscr{P}^D_0 \tag{5.12}$$

of preorders and monotone maps. We will verify first that it satisfies Conditions (1) and (2) of Theorem 5.6.

(1) Each $\mathscr{P}^D_n$ contains a nonempty finite final subset since the module M is assumed to be compact.

(2) The image of every upward-closed set under the monotone map p^{n+1}_n is upward-closed. Denote the value of $D : \mathscr{D} \longrightarrow \mathsf{Complex}(M)$ by

$$\begin{array}{c} Dd \\ \big\downarrow {\scriptstyle D\delta} \\ Dd' \end{array} \quad = \quad \begin{array}{ccccccc} \cdots & \xrightarrow{m^d_3} & a^d_2 & \xrightarrow{m^d_2} & a^d_1 & \xrightarrow{m^d_1} & a^d_0 \\ & & \big\downarrow {\scriptstyle \delta_2} & & \big\downarrow {\scriptstyle \delta_1} & & \big\downarrow {\scriptstyle \delta_0} \\ \cdots & \xrightarrow[m^{d'}_3]{} & a^{d'}_2 & \xrightarrow[m^{d'}_2]{} & a^{d'}_1 & \xrightarrow[m^{d'}_1]{} & a^{d'}_0 \end{array}$$

Then the value of $\mathrm{pr}_n \cdot D : \mathscr{D} \longrightarrow \mathsf{Complex}_n(M)$ is given by

$$\begin{array}{c} \mathrm{pr}_n \cdot Dd \\ \big\downarrow {\scriptstyle \mathrm{pr}_n \cdot D\delta} \\ \mathrm{pr}_n \cdot Dd' \end{array} \quad = \quad \begin{array}{ccccccccc} a^d_n & \xrightarrow{m^d_n} & \cdots & \xrightarrow{m^d_3} & a^d_2 & \xrightarrow{m^d_2} & a^d_1 & \xrightarrow{m^d_1} & a^d_0 \\ \big\downarrow {\scriptstyle \delta_n} & & & & \big\downarrow {\scriptstyle \delta_2} & & \big\downarrow {\scriptstyle \delta_1} & & \big\downarrow {\scriptstyle \delta_0} \\ a^{d'}_n & \xrightarrow[m^{d'}_n]{} & \cdots & \xrightarrow[m^{d'}_3]{} & a^{d'}_2 & \xrightarrow[m^{d'}_2]{} & a^{d'}_1 & \xrightarrow[m^{d'}_1]{} & a^{d'}_0 \end{array}$$

for every $n \geq 0$.

Choose an upward-closed set $S \subseteq \mathscr{P}^D_{n+1}$. Every $s \in S$ is a cone for the above diagram $\mathrm{pr}_{n+1} \cdot D$ and we denote this cone by

$$s = \begin{array}{ccccccccc} s_{n+1} & \xrightarrow{m^s_{n+1}} & \cdots & \xrightarrow{m^s_3} & s_2 & \xrightarrow{m^s_2} & s_1 & \xrightarrow{m^s_1} & s_0 \\ \big\downarrow {\scriptstyle \sigma^d_{n+1}} & & & & \big\downarrow {\scriptstyle \sigma^d_2} & & \big\downarrow {\scriptstyle \sigma^d_1} & & \big\downarrow {\scriptstyle \sigma^d_0} \\ a^d_{n+1} & \xrightarrow[m^d_{n+1}]{} & \cdots & \xrightarrow[m^d_3]{} & a^d_2 & \xrightarrow[m^d_2]{} & a^d_1 & \xrightarrow[m^d_1]{} & a^d_0 \end{array}$$

Choose any s in S and consider b in $\mathscr{P}_n^D$ such that $p_n^{n+1}(s) \sqsubseteq_n b$ holds. We need to find $s \sqsubseteq_{n+1} t$ such that $p_n^{n+1}(t) = b$.
In our notation, b has the form

$$b = \begin{array}{ccccccccc} b_n & \xrightarrow{m_n^b} & \cdots & \xrightarrow{m_3^b} & b_2 & \xrightarrow{m_2^b} & b_1 & \xrightarrow{m_1^b} & b_0 \\ \downarrow{\scriptstyle \beta_n^d} & & & & \downarrow{\scriptstyle \beta_2^d} & & \downarrow{\scriptstyle \beta_1^d} & & \downarrow{\scriptstyle \beta_0^d} \\ a_n^d & \xrightarrow[m_n^d]{} & \cdots & \xrightarrow[m_3^d]{} & a_2^d & \xrightarrow[m_2^d]{} & a_1^d & \xrightarrow[m_1^d]{} & a_0^d \end{array}$$

The inequality $p_n^{n+1}(s) \sqsubseteq_n b$ means that there exists a diagram of the form

$$\begin{array}{ccccccccc} s_n & \xrightarrow{m_n^s} & \cdots & \xrightarrow{m_3^s} & s_2 & \xrightarrow{m_2^s} & s_1 & \xrightarrow{m_1^s} & s_0 \\ \downarrow{\scriptstyle g_n} & & & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} \\ b_n & \xrightarrow{m_n^b} & \cdots & \xrightarrow{m_3^b} & b_2 & \xrightarrow{m_2^b} & b_1 & \xrightarrow{m_1^b} & b_0 \\ \downarrow{\scriptstyle \beta_n^d} & & & & \downarrow{\scriptstyle \beta_2^d} & & \downarrow{\scriptstyle \beta_1^d} & & \downarrow{\scriptstyle \beta_0^d} \\ a_n^d & \xrightarrow[m_n^d]{} & \cdots & \xrightarrow[m_3^d]{} & a_2^d & \xrightarrow[m_2^d]{} & a_1^d & \xrightarrow[m_1^d]{} & a_0^d \end{array}$$

where the equalities $\beta_i^d \cdot g_i = \sigma_i^d$ hold for every $i \in \{0, \dots, n\}$.
Consider the following diagram:

$$\begin{array}{ccccccccccc} s_{n+1} & \xrightarrow{m_{n+1}^s} & s_n & \xrightarrow{m_n^s} & \cdots & \xrightarrow{m_3^s} & s_2 & \xrightarrow{m_2^s} & s_1 & \xrightarrow{m_1^s} & s_0 \\ \| & & \downarrow{\scriptstyle g_n} & & & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} \\ s_{n+1} & \xrightarrow{g_n @ m_{n+1}^s} & b_n & \xrightarrow{m_n^b} & \cdots & \xrightarrow{m_3^b} & b_2 & \xrightarrow{m_2^b} & b_1 & \xrightarrow{m_1^b} & b_0 \\ \downarrow{\scriptstyle \beta_{n+1}^d} & & \downarrow{\scriptstyle \beta_n^d} & & & & \downarrow{\scriptstyle \beta_2^d} & & \downarrow{\scriptstyle \beta_1^d} & & \downarrow{\scriptstyle \beta_0^d} \\ a_{n+1}^d & \xrightarrow[m_{n+1}^d]{} & a_n^d & \xrightarrow[m_n^d]{} & \cdots & \xrightarrow[m_3^d]{} & a_2^d & \xrightarrow[m_2^d]{} & a_1^d & \xrightarrow[m_1^d]{} & a_0^d \end{array}$$

Thus, the desired t has the form

$$\begin{array}{ccccccccccc} s_{n+1} & \xrightarrow{g_n @ m_{n+1}^s} & b_n & \xrightarrow{m_n^b} & \cdots & \xrightarrow{m_3^b} & b_2 & \xrightarrow{m_2^b} & b_1 & \xrightarrow{m_1^b} & b_0 \\ \downarrow{\scriptstyle \beta_{n+1}^d} & & \downarrow{\scriptstyle \beta_n^d} & & & & \downarrow{\scriptstyle \beta_2^d} & & \downarrow{\scriptstyle \beta_1^d} & & \downarrow{\scriptstyle \beta_0^d} \\ a_{n+1}^d & \xrightarrow[m_{n+1}^d]{} & a_n^d & \xrightarrow[m_n^d]{} & \cdots & \xrightarrow[m_3^d]{} & a_2^d & \xrightarrow[m_2^d]{} & a_1^d & \xrightarrow[m_1^d]{} & a_0^d \end{array}$$

Hence the image of every upward-closed set under the monotone map $p_n^{n+1} : \mathscr{P}_{n+1}^D \longrightarrow \mathscr{P}_n^D$ is upward-closed.

Therefore, by Theorem 5.6, we have an element (x_n) of the limit $\lim \mathscr{P}_n^D$.

Denote every x_n as follows:

$$x_n = \begin{array}{ccccccccc} x^n_n & \xrightarrow{m^{x_n}_n} & \cdots & \xrightarrow{m^{x_n}_3} & x^n_2 & \xrightarrow{m^{x_n}_2} & x^n_1 & \xrightarrow{m^{x_n}_1} & x^n_0 \\ \downarrow{\scriptstyle \chi^{n,d}_n} & & & & \downarrow{\scriptstyle \chi^{n,d}_2} & & \downarrow{\scriptstyle \chi^{n,d}_1} & & \downarrow{\scriptstyle \chi^{n,d}_0} \\ a^d_n & \xrightarrow[m^d_n]{} & \cdots & \xrightarrow[m^d_3]{} & a^d_2 & \xrightarrow[m^d_2]{} & a^d_1 & \xrightarrow[m^d_1]{} & a^d_0 \end{array}$$

From that we can define a complex

$$\cdots \xrightarrow{m^{x_4}_4} x^3_3 \xrightarrow{m^{x_3}_3} x^2_2 \xrightarrow{m^{x_2}_2} x^1_1 \xrightarrow{m^{x_1}_1} x^0_0$$

that is obviously a vertex of a cone for $D : \mathscr{D} \longrightarrow \mathsf{Complex}(M)$. □

Corollary 5.16. Every compact module satisfying the Weak Solvability Condition has a final coalgebra.

Corollary 5.17. Every finitary endofunctor of the category of linear orders and order-preserving maps has a final coalgebra.

Example 5.18. Recall from Example 5.14 that the modules corresponding to the finitary endofunctors

$$X \mapsto X * \omega, \quad X \mapsto (X * \omega)\ ;\ 1$$

of the category Lin are compact. Since Lin satisfies the Weak Solvability Conditions, the above two functors have final coalgebras by the above corollary. The linear orders of these coalgebras are the continuum and Cantor space, respectively, see (Pavlović and Pratt (2002)) for a proof.

Remark 5.19. T. Leinster studies final coalgebras on categories that are formed of objects that (when represented as set-valued functors) are sums of flat functors. Such categories are precisely the iterated cocompletions Fam(Ind$\mathscr{A}$), first under filtered colimits, then under sums of small categories. We know from (Makkai and Paré (1989)), 5.3.2, that such categories are equivalent to Ind(fam$\mathscr{A}$), cocompletions of small categories under first (finite) coproducts, then filtered colimits. Moreover the functors on them that are studied in (Leinster (2011)) preserve all connected colimits, in particular the filtered ones. Hence the framework of T. Leinster is subsumed under ours.

Remark 5.20. Our results can be extended to finitary endofunctors on categories that are "continuous" in the sense of (Johnstone and Joyal (1982)), with a small, full, dense and suitably filtered subcategory. Such categories $\mathscr{E}$ arise as retracts, by filtered colimit preserving functors, of finitely accessible ones:

$$\mathscr{E} \underset{\iota}{\overset{\rho}{\rightleftarrows}} \mathrm{Ind}(\mathscr{C})$$

Then the existence of a final coalgebra for a finitary endofunctor Φ on $\mathscr{E}$ can be reduced to the existence of a final coalgebra for $\iota \cdot \Phi \cdot \rho$ on the finitely accessible category Ind($\mathscr{C}$).

6. What the Existence of a Final Coalgebra Entails

We show in this section that the existence of final coalgebras entails the Weak Solvability Condition, provided the module is pointed. As a corollary, we derive a necessary condition on the category $\mathscr{A}$ so that the identity functor on $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ admits a final coalgebra, see Corollary 6.4.

Assumption 6.1. We assume in this section that M is *pointed*, i.e., that M is equipped with a module morphism $c : \mathscr{A} \longrightarrow M$.

Of course, the assumption is clearly satisfied if and only if, when passing from M to the finitary endofunctor Φ, there exists a natural transformation $\eta\colon \mathrm{Id} \longrightarrow \Phi$. This is the case when Φ arises as $\Phi = G \cdot F$, where F is a left adjoint, G is a right adjoint preserving filtered colimits and η is the unit of the adjunction. For example, consider the endofunctor Φ on $\mathsf{Pos}_{0,1}$ that assigns to a poset with distinct endpoints $(X, 0, 1)$ the set of finitely based lower sets $(\{\downarrow(x_1, ..., x_n) | x_i \in X\}, \{0\}, \downarrow(1))$ ordered by inclusion. This functor takes really values in the category of join-semilattices with distinct endpoints, so Φ as an endofunctor of $\mathsf{Pos}_{0,1}$ arises as a composite with the forgetful functor from join-semilattices with distinct endpoints. This endofunctor does not admit a final coalgebra and this is not coincidental as the next Theorem shows.

Remark 6.2. From Assumption 6.1 it follows that every representable functor $\mathscr{A}(a, -)$ admits a coalgebra structure

$$c_a : \mathscr{A}(a, -) \longrightarrow M(a, -)$$

for $M \otimes -$ (we used that $\Big(M \otimes \mathscr{A}\Big)(a, -) \cong M(a, -)$ holds). This of course entails that $\mathsf{Complex}(M)$ is nonempty, see Proposition 3.6.

Moreover, for every $f : a \longrightarrow a'$, the natural transformation $\mathscr{A}(f, -) : \mathscr{A}(a', -) \longrightarrow \mathscr{A}(a, -)$ is a coalgebra morphism, i.e., the square

$$\begin{array}{ccc} \mathscr{A}(a', -) & \xrightarrow{\;c_{a'}\;} & M(a', -) \\ {\scriptstyle \mathscr{A}(f,-)}\downarrow & & \downarrow{\scriptstyle M(f,-)} \\ \mathscr{A}(a, -) & \xrightarrow[\;c_a\;]{} & M(a, -) \end{array} \tag{6.13}$$

commutes.

Theorem 6.3. Suppose that M is pointed and suppose that a final coalgebra for $M \otimes -$ exists. Then pr_0 is cofiltering, i.e., the Weak Solvability Condition holds.

Proof. Let us denote by $j : J \longrightarrow M \otimes J$ the final coalgebra for $M \otimes -$.

Denote by $c_a^\dagger : \mathscr{A}(a, -) \longrightarrow J$ the unique coalgebra morphism such that the square

$$\begin{array}{ccc} \mathscr{A}(a, -) & \xrightarrow{\;c_a\;} & M \otimes \mathscr{A}(a, -) \\ {\scriptstyle c_a^\dagger}\downarrow & & \downarrow{\scriptstyle M \circ c_a^\dagger} \\ J & \xrightarrow[\;j\;]{} & M \otimes J \end{array}$$

commutes.

Then the following triangle

$$\begin{array}{ccc} \mathscr{A}(a',-) & \xrightarrow{c^{\dagger}_{a'}} & \\ {\scriptstyle \mathscr{A}(f,-)}\downarrow & & J \\ \mathscr{A}(a,-) & \xrightarrow{c^{\dagger}_{a}} & \end{array}$$

commutes by finality of $j : J \longrightarrow M \otimes J$ and the square (6.13).

Recall that, in any case, one can form a colimit I of the diagram

$$\Big(\mathsf{Complex}(M)\Big)^{op} \xrightarrow{\mathrm{pr}_0^{op}} \mathscr{A}^{op} \xrightarrow{Y} [\mathscr{A}, \mathsf{Set}]$$

We do not claim that $I : \mathscr{A} \longrightarrow \mathsf{Set}$ is flat. In fact, we will just use the fact that I is a colimit. For observe that so far we have proved that the collection of morphisms

$$c^{\dagger}_{\mathrm{pr}_0^{op}(a_\bullet, m_\bullet)} : \mathscr{A}(a_0, -) \longrightarrow J$$

forms a cocone for the diagram $Y \cdot \mathrm{pr}_0^{op}$. Hence there exists a natural transformation

$$\overline{\beta} : I \longrightarrow J$$

The natural transformation $\overline{\beta}$ induces a functor $F : \mathsf{Complex}(M) \longrightarrow \mathsf{elts}(J)$ by putting

$$(a_\bullet, m_\bullet) \mapsto x \in Ja_0$$

where the element $x \in Ja_0$ corresponds to the natural transformation $c^{\dagger}_{\mathrm{pr}_0^{op}(a_\bullet, m_\bullet)} : \mathscr{A}(a_0, -) \longrightarrow J$ by Yoneda Lemma.

Then the diagram

$$\begin{array}{ccc} \mathsf{Complex}(M) & \xrightarrow{F} & \mathsf{elts}(J) \\ & {\scriptstyle \mathrm{pr}_0}\searrow \quad \swarrow{\scriptstyle \mathsf{proj}} & \\ & \mathscr{A} & \end{array}$$

commutes. Since J is a flat functor, the category $\mathsf{elts}(J)$ is cofiltered. Hence $\mathrm{pr}_0 = \mathsf{proj} \cdot F$ is a cofiltering functor. □

Corollary 6.4. If the identity functor on the category $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ has a final coalgebra, then the category $\mathscr{A}$ must be cofiltered.

Remark 6.5. The above Corollary shows that the identity endofunctor of a Scott complete category $\mathscr{K}$, see Example 2.4((6)), canot have a final coalgebra unless the category $\mathscr{K}$ is in fact locally finitely presentable.

What we have proved so far, allows us to go in full circle:

Corollary 6.6. Suppose that $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{A}$ is a pointed, compact module. Then the following are equivalent:

(1) The self-similarity system $(\mathscr{A}, M)$ satisfies the Weak Solvability Condition.

(2) The self-similarity system $(\mathscr{A}, M)$ satisfies the Strong Solvability Condition.
(3) The colimit of the diagram

$$\Big(\mathsf{Complex}(M)\Big)^{op} \xrightarrow{\mathrm{pr}_0^{op}} \mathscr{A}^{op} \xrightarrow{Y} [\mathscr{A}, \mathsf{Set}]$$

is a flat functor.
(4) The final coalgebra for $M \otimes -$ exists.

7. Conclusions and Future Research

We have provided a new uniform way of constructing final coalgebras for finitary endofunctors of locally finitely presentable categories. We have argued about the necessity of expanding these results to the case of finitely accessible categories. To that end we have formulated general conditions that are sufficient for the existence of a final coalgebra. We expect that our conditions can be exploited for finding new interesting examples of final coalgebras in accessible categories.

In many concrete examples where the final coalgebra cannot exist for cardinality reasons (e.g., the categories where all maps are injections) we expect that suitable modifications of our results will provide coalgebras of "rational terms". This means coalgebras comprising of solutions of finitary recursive systems, see (Adámek, Milius and Velebil (2006)).

A further extension of our work, based on a suitable modification of the key-notion of complex, is expected to yield results about the existence of cofree coalgebras in the environment of finitely accessible categories.

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