COFREE COALGEBRAS ON ACCESSIBLE CATEGORIES

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Abstract. Given a finitary functor $\Phi : \mathscr{K} \to \mathscr{K}$ and an object K of $\mathscr{K}$, where $\mathscr{K}$ is a finitely accessible category, we construct (under certain conditions) the cofree Φ-coalgebra on K. The conditions are always satisfied when $\mathscr{K}$ is locally finitely presentable or, more generally, a finitely accessible category where consistent diagrams have colimits (=Scott-complete category) and they yield a right-adjoint $C \vdash U$, for the forgetful functor $U : \mathrm{Coalg}(\Phi) \to \mathscr{K}$ from the category of Φ-coalgebras to $\mathscr{K}$. Our construction gives as a special case that of a final coalgebra for finitary endofunctors on locally finitely presentable categories, as described in previous work of ours, and relies on a modification of the notion of complex (of modular elements) introduced by T. Leinster.

1. Introduction

The concept of coalgebra has become, during the past decade, the domain of extensive research in theoretical computer science, in connection with the study of automata and infinite data types. Much of this research is directed towards answering the questions of existence and identification in concrete terms of final coalgebras, as those that codify the key notion of bisimulation between processes. In the same vein, among the questions studied is that of the existence of cofree coalgebras, in view of providing a structural characterization of categories of coalgebras dual to that of Birkhoff's characterization of varieties of algebras (see [R], Thm. 17.3, and [V], Thm. 14.4).

Whereas much of the initial work in the field was revolving around the notion of coalgebra for an endofunctor of the category of sets, it has gradually become apparent that we need to move towards the study of coalgebras over more general categories. For example, various categories of (partially) ordered sets provide the necessary framework for a coalgebraic account of classical structures such as the real closed interval or the Baire space ([F], [PP]). Other general categories accommodate various structures of fractal nature as final coalgebras, as shown in [Le_1]. There Tom Leinster introduces the fundamental for our purposes technique of complexes of modular elements as a general method of construction of final coalgebras. At the same time suitable general categories provide adequate background for extending coalgebraic logic, i.e the characterization of various classes of coalgebras in terms of satisfaction of formulae in modal logic ([Kl]), as well as for extending structural properties of final coalgebras (e.g as completions of initial algebras [A_1]). The common feature of these underlying categories, that makes such generalizations possible, is that they form examples (or special subclasses) of finitely accessible categories.

In this work we construct the cofree Φ-coalgebra on an object K of a finitely accessible category $\mathscr{K}$, when Φ is a finitary endofunctor and a certain further condition applies. In particular our condition always holds when the underlying category $\mathscr{K}$ is locally finitely presentable (l.f.p). This way we obtain a concrete description of the cofree coalgebra, which is known to exist in this case as a corollary to the existence of bilimits in the category of accessible categories (in combination with the fact that a category of coalgebras is an inserter in this 2-category). Specializing our construction to the cofree coalgebra over the terminal object of the underlying l.f.p category we obtain a construction of the final coalgebra that applies uniformly to all finitary endofunctors of l.f.p categories (the final coalgebra is also known to exist in this case for the 2-categorical reasons explained above). As said, the mere existence of cofree coalgebras and of final coalgebras is well-known in the case of l.f.p categories. Moreover, the former question is reducible to the latter, as a cofree coalgebra on the object C, for the endofunctor Φ of a category with products, is simply the final coalgebra for the endofunctor $C \times \Phi(-)$. The applicability of the notion of complexes of modular elements (due to Tom Leinster) to the construction of the final coalgebra over l.f.p categories (but also to other finitely accessible categories) is discussed in some earlier work of ours ([KMV]).

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When it comes to the realm of l.f.p categories, the novelty of this work lies in the fact that our results give a self-contained account of the construction of final and cofree coalgebras, that circumvents certain technicalities of [Le_1]. But furthermore our extra condition applies to all finitary endofunctors of *Scott complete categories* ([A_2]), i.e finitely accessible categories where all diagrams that have a cocone, have a colimit. Such categories may lack a terminal object, so the existence of the final coalgebra for a finitary endofunctor is not granted. Nevertheless the cofree coalgebra over an arbitrary object of the underlying category can always be constructed. A prominent example of a Scott complete category is that of partially ordered sets with distinct endpoints (and non-decreasing maps preserving the endpoints), used in Peter Freyd's construction of the real closed interval as a final coalgebra (for a suitable finitary endofunctor). More generally, elaborating on a syntactical description of such categories in [A_2], we can think of such categories as categories of models for limit theories, extended by axioms declaring the distinctness of closed terms (e.g $0 \neq 1$, in the aforementioned example).

Organization of the paper. Our construction relies on a modification of the notion of *complex of modular elements*, developed by T. Leinster ([Le_1]): Finitely accessible categories are, up to equivalence, categories of the form $\mathscr{K} \simeq \mathrm{Flat}(\mathscr{A}, \mathrm{Set})$ of flat functors from a small category $\mathscr{A}$ to the category of sets. The small category $\mathscr{A}$ is, up to splitting of idempotents, equivalent to $\mathscr{K}_{fp}^{op}$, the opposite of the full subcategory of finitely presentable objects of $\mathscr{K}$. Finitary endofunctors Φ of a finitely accessible category $\mathscr{K}$ correspond to certain bimodules $M_\Phi\colon \mathscr{K}_{fp} \times \mathscr{K}_{fp}^{op} \longrightarrow \mathsf{Set}$. The complexes are then, roughly, formal countable composites of elements of the bimodule. They form a category that parametrizes a suitable colimit, which yields the final coalgebra for Φ. Here we consider instead *pointed complexes*, which are, roughly again, similar formal countable composites of modular elements between pairs consisting of an object $a \in \mathscr{K}_{fp}$ and an element $x \in Ka$ of the object K (seen as a functor under the above equivalence), over which the cofree coalgebra is constructed. We present all the relevant technical details in Section 2.

A colimit parametrized by the category of pointed complexes yields the (carrier of the) cofree coalgebra over K for Φ. Our main technical result is that the cofree coalgebra over K exists, provided that the category of K-pointed complexes is co-filtered (so that the carrier of the cofree coalgebra exists as a filtered colimit of finitely presentable objects in the category $\mathscr{K}$). The details of the (lengthy) proof occupy Section 3.

In Section 4. we discuss Scott-complete categories and show that the cofilteredness of the category of pointed complexes is always satisfied for finitary endofunctors on them.

2. Preliminaries

Coalgebras and final coalgebras. We give a precise definition of (final) coalgebras. See, e.g., [R] for motivation and examples of various coalgebras in the category of sets.

Definition 2.1. Suppose $\Phi : \mathscr{K} \longrightarrow \mathscr{K}$ is any functor.

(1) A *coalgebra for* Φ is a morphism $e : X \longrightarrow \Phi(X)$.

(2) A *homomorphism of coalgebras* from $e : X \longrightarrow \Phi(X)$ to $e' : X' \longrightarrow \Phi(X')$ is a morphism $h : X \longrightarrow X'$ making the following square

$$\begin{array}{ccc} X & \xrightarrow{e} & \Phi(X) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Phi(h)} \\ X' & \xrightarrow[e']{} & \Phi(X') \end{array}$$

commutative.

(3) A coalgebra $\tau : T \longrightarrow \Phi(T)$ is called *final*, if it is a terminal object of the category of coalgebras, i.e., if for every coalgebra $e : X \longrightarrow \Phi(X)$ there is a unique morphism $e^\dagger : X \longrightarrow T$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{e} & \Phi(X) \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle \Phi(e^\dagger)} \\ T & \xrightarrow[\tau]{} & \Phi(T) \end{array}$$

commutes.

Finitely accessible and locally finitely presentable categories. Finitely accessible and locally finitely presentable categories are those where every object can be reconstructed knowing its "finite parts". This is a property that, for example, the category Set of sets and mappings has, where a set P is recognized as finite exactly when its hom-functor $\mathsf{Set}(P, -) : \mathsf{Set} \longrightarrow \mathsf{Set}$ preserves colimits of a certain class — the so-called *filtered* colimits. We can also verify this property for the category Pos of partially oredered sets and order-preserving mappings, or for the category Lin of linearly oredered sets and order-preserving mappings. We can also verify it for any category of algebras defined by finitary operations and homomorphisms between them (there the "finite" objects are the finitely presentable ones, in the classical sense of algebra), but not for, say, the category of topological spaces and continuous mappings.

A colimit of a general diagram $D : \mathscr{D} \longrightarrow \mathscr{K}$ is called *filtered*, provided that its scheme-category $\mathscr{D}$ is filtered. A category $\mathscr{D}$ is called *filtered* provided that every finite subcategory of $\mathscr{D}$ admits a cocone. In more elementary terms, filteredness of $\mathscr{D}$ can be expressed equivalently by the following three properties:

(1) The category $\mathscr{D}$ is nonempty.
(2) Each pair d_1, d_2 of objects of $\mathscr{D}$ has an "upper bound", i.e., there exists a cocone

$$\begin{array}{ccc} d_1 & \searrow & \\ & & d \\ d_2 & \nearrow & \end{array}$$

in $\mathscr{D}$.
(3) Each parallel pair of morphisms in $\mathscr{D}$ can be "coequalized", i.e., for each parallel pair

$$d_1 \rightrightarrows d_2$$

of morphisms in $\mathscr{D}$ there is a completion to a commutative diagram of the form

$$d_1 \rightrightarrows d_2 \longrightarrow d$$

in $\mathscr{D}$.

A category is $\mathscr{D}$ called *cofiltered* provided that the dual category $\mathscr{D}^{op}$ is filtered.

An object P of a category $\mathscr{K}$ is called *finitely presentable* if the hom-functor $\mathscr{K}(P, -) : \mathscr{K} \longrightarrow \mathsf{Set}$ preserves filtered colimits.

Definition 2.2. A category $\mathscr{K}$ is called *finitely accessible* if it has filtered colimits and if it contains a small subcategory consisting of finitely presentable objects such that every object of $\mathscr{K}$ is a filtered colimit of these finitely presentable objects.

A cocomplete finitely accessible category is called *locally finitely presentable*.

Example 2.3.

(1) The category Set of sets and mappings is locally finitely presentable. The finitely presentable objects are exactly the finite sets.
(2) Every variety of finitary algebras is a locally finitely presentable category. The finitely presentable objects are exactly the algebras that are presented by finitely many generators and finitely many equations in the sense of universal algebra.
(3) The category Inj having sets as objects and injective maps as morphisms is a finitely accessible category that is not locally finitely presentable. The finitely presentable objects are exactly the finite sets.
(4) Denote by Field the category of fields and field homomorphisms. Then Field is a finitely accessible category that is not locally finitely presentable.
(5) The category Lin of linear orders and monotone maps is finitely accessible but not locally finitely presentable. The finitely presentable objects are exactly the finite ordinals.
(6) Let $\mathsf{Pos}_{0,1}$ denote the following category:
    (a) Objects are posets having distinct top and bottom elements.
    (b) Morphisms are monotone maps preserving top and bottom elements.

    Then $\mathsf{Pos}_{0,1}$ is a *Scott complete* category in the sense of Jiří Adámek [A_2]: it is finitely accessible and every small diagram in $\mathsf{Pos}_{0,1}$ that has a cocone, has a colimit.

    Scott complete categories are therefore "not far away" from being cocomplete and thus locally finitely presentable.

    However, $\mathsf{Pos}_{0,1}$ is not locally finitely presentable since it lacks a terminal object. Finitely presentable objects in $\mathsf{Pos}_{0,1}$ are exactly the finite posets having distinct bottom and top elements.
(7) The category of topological spaces and continuous maps is not finitely accessible. Although this category has filtered (in fact, all) colimits, the only finitely presentable objects are finite discrete topological spaces and these do not suffice for reconstruction of a general topological space.

Flat functors. Every finitely accessible category $\mathscr{K}$ is equivalent to a category of the form

$$\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$$

(where $\mathscr{A}$ is a small category) that consists of all *flat functors* $X : \mathscr{A} \longrightarrow \mathsf{Set}$ and all natural transformations between them. The small category $\mathscr{A}$ is, up to splitting of idempotents, equivalent to $\mathscr{K}_{fp}^{op}$, the opposite of the full subcategory of finitely presentable objects of $\mathscr{K}$.

A functor $X : \mathscr{A} \longrightarrow \mathsf{Set}$ is called *flat* if its *category of elements* $\mathsf{elts}(X)$ is cofiltered. The category $\mathsf{elts}(X)$ has pairs (x, a) with $x \in Xa$ as objects and as morphisms from (x, a) to (x', a') those morphisms $f : a \longrightarrow a'$ in $\mathscr{A}$ with the property that $Xf(x) = x'$.

Flat functors X can be characterized by any of the following equivalent conditions:

1. The functor $X : \mathscr{A} \longrightarrow \mathsf{Set}$ is a filtered colimit of representable functors.
2. The left Kan extension $\mathrm{Lan}_Y X : [\mathscr{A}^{op}, \mathsf{Set}] \longrightarrow \mathsf{Set}$ of $X : \mathscr{A} \longrightarrow \mathsf{Set}$ along the Yoneda embedding $Y : \mathscr{A} \longrightarrow [\mathscr{A}^{op}, \mathsf{Set}]$ preserves finite limits.

In case when $\mathscr{K}$ is locally finitely presentable one can prove that $\mathscr{K}$ is equivalent to the category

$$\mathsf{Lex}(\mathscr{A}, \mathsf{Set})$$

of all finite-limits-preserving functors on a small finitely complete category $\mathscr{A}$. In fact, the flat functors are exactly the finite-limits-preserving ones in this case.

Remark 2.4. The category $\mathscr{A}$ can be chosen as the dual category of finitely presentable objects, $\mathscr{K}_{fp}^{op}$.

Flat modules. On finitely accessible categories there is class of functors that can be fully reconstructed by knowing their values on "finite parts". An example is the finite-powerset endofunctor

$$P_{fin} : X \mapsto \{S \mid S \subseteq X,\ S \text{ is finite }\}$$

of the category of sets. Such endofunctors can be characterized as exactly those *preserving* filtered colimits.

Definition 2.5. A functor $\Phi : \mathscr{K} \longrightarrow \mathscr{L}$ between finitely accessible categories is called *finitary* if it preserves filtered colimits.

By the above considerations, every finitary endofunctor $\Phi : \mathscr{K} \longrightarrow \mathscr{K}$ of a finitely accessible category $\mathscr{K}$ can be considered, to within equivalence, as a finitary endofunctor

$$\Phi : \mathsf{Flat}(\mathscr{A}, \mathsf{Set}) \longrightarrow \mathsf{Flat}(\mathscr{A}, \mathsf{Set})$$

Since the full embedding $\mathscr{A}^{op} \longrightarrow \mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ exhibits $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ as a free cocompletion of $\mathscr{A}^{op}$ w.r.t. filtered colimits (also denoted as $\mathrm{Ind}(\mathscr{A})$, the "inductive" cocompletion), we can then reconstruct Φ from a mere functor

$$M_\Phi : \mathscr{A}^{op} \longrightarrow \mathsf{Flat}(\mathscr{A}, \mathsf{Set})$$

(no preservation properties) by means of filtered colimits.

The latter functor can be identified with a functor of the form $M_\Phi : \mathscr{A}^{op} \times \mathscr{A} \longrightarrow \mathsf{Set}$ with the property that every $M_\Phi(a, -) : \mathscr{A} \longrightarrow \mathsf{Set}$ is flat. Such functors of two variables (without the extra flatness property) are commonly called *modules*. We will give the extra property a name.

Definition 2.6. A *module* $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathscr{B}$ from a small category $\mathscr{A}$ to a small category $\mathscr{B}$ is a functor $M : \mathscr{A}^{op} \times \mathscr{B} \longrightarrow \mathsf{Set}$. Given two such modules, M and N, a *module morphism* $M \longrightarrow N$ is a natural transformation between the respective functors.

A module M as above is called *flat* if every partial functor $M(a, -) : \mathscr{B} \longrightarrow \mathsf{Set}$ is a flat functor in the usual sense.

Remark 2.7. The above module terminology makes perfect sense if we denote an element $m \in M(a, b)$ by an arrow

$$a \xrightarrow{m} b$$

and think of it as of a "vector" on which the categories $\mathscr{A}$ and $\mathscr{B}$ can act by means of their morphisms ("scalars"):

(1) Given $f : a' \longrightarrow a$ in $\mathscr{A}$, then

$$a' \xrightarrow{f} a \xrightarrow{m} b$$

denotes the element $M(f, b)(m) \in M(a', b)$.

Had we denoted such an action by $m@f$, then it is obvious that equations $m@(f \cdot f') = (m@f)@f'$ and $m@1_a = m$ hold — something that we know from classical module theory.

(2) Given $g : b \longrightarrow b'$ in $\mathscr{B}$, then

$$a \overset{m}{\longrightarrow\!\!\!\!\!\!\!+\;\;} b \overset{g}{\longrightarrow} b'$$

denotes the element $M(a, g)(m) \in M(a, b')$.

(3) Functoriality of M gives an unambiguous meaning to diagrams of the form

$$a' \overset{f}{\longrightarrow} a \overset{m}{\longrightarrow\!\!\!\!\!\!\!+\;\;} b \overset{g}{\longrightarrow} b'$$

(4) We also extend the notion of commutative diagrams. For example, by saying that the following square

$$\begin{array}{ccc} a & \overset{m}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & b \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ a' & \underset{m'}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & b' \end{array}$$

commutes we mean that the equality $m'@f = g@m$ holds.

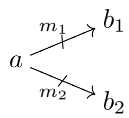
Remark 2.8. The broken arrow notation also allows us to formulate flatness of a module $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{B}$ in elementary terms. Namely, for every a in $\mathscr{A}$ the following three conditions must be satisfied:

(1) There is a broken arrow

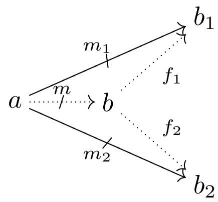
$$a \overset{m}{\longrightarrow\!\!\!\!\!\!\!+\;\;} b$$

for some b in $\mathscr{B}$.

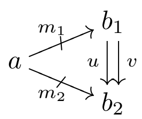
(2) For any two broken arrows



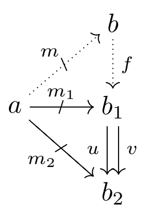
there is a commutative diagram



(3) For every commutative diagram



there is a commutative diagram



Definition 2.9. Suppose $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{B}$ and $N : \mathscr{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{C}$ are modules. By

$$N \otimes M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathscr{C}$$

we denote their *composition* which is defined objectwise by means of a coend

$$\Big(N \otimes M\Big)(a, c) = \int^{b} N(b, c) \times M(a, b)$$

Remark 2.10. A coend is a special kind of colimit. The elements of $\Big(N \otimes M\Big)(a,c)$ are equivalence classes. A typical element of $\Big(N \otimes M\Big)(a,c)$ is an equivalence class $[(n,m)]$ represented by a pair $(n,m) \in N(b,c) \times M(a,b)$ where the equivalence is generated by requiring the pairs

$$(n, f@m) \quad \text{and} \quad (n@f, m)$$

to be equivalent, where n, f and m are as follows:

$$a \xrightarrow{m} b \xrightarrow{f} b' \xrightarrow{n} c$$

Above, we denoted the actions of M and N by the same symbols, not to make the notation heavy.

It is well-known (see [Bo]) that the above composition organizes modules into a *bicategory*: the composition is associative only up to a coherent isomorphism and the *identity module* $\mathscr{A} : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!\mapsto \mathscr{A}$, sending (a', a) to the hom-set $\mathscr{A}(a', a)$, serves as a unit only up to a coherent isomorphism. The following result is then easy to prove.

Lemma 2.11. *Every identity module is flat and composition of flat modules is a flat module.*

Remark 2.12. The above composition of modules makes one to attempt to draw diagrams such as

$$a_2 \xrightarrow{m_2} a_1 \xrightarrow{m_1} a_0$$

for elements $m_1 \in M(a_1, a_0)$, $m_2 \in M(a_2, a_1)$ of a module $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!\mapsto \mathscr{A}$. Such diagrams are, however, to be considered only formally — *we never compose two "broken" arrows*.

The tensor notation from the above paragraphs allows us to pass from endofunctors to modules completely.
Observe that any flat functor $X : \mathscr{A} \longrightarrow \mathsf{Set}$ can be considered as a flat module $X : \mathbf{1} \longrightarrow\!\!\!\!\!\!\!\mapsto \mathscr{A}$ where $\mathbf{1}$ denotes the one-morphism category.
Then, given a flat module $M : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!\mapsto \mathscr{A}$, the assignment $X \mapsto M \otimes X$ defines a finitary endofunctor of $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$.
In fact, every finitary endofunctor Φ of $\mathsf{Flat}(\mathscr{A}, \mathsf{Set})$ arises in the above way: construct the flat module M_Φ as above, then there is an isomorphism

$$\Phi \cong M_\Phi \otimes -$$

of functors.

Assumption 2.13. In the rest of the paper,

$$\mathscr{K} \cong \mathsf{Flat}(\mathscr{A}, \mathsf{Set})$$

denotes a finitely accessible category, $\mathscr{A}$ is the dual of the category $\mathscr{K}_{fp}$ representing finitely presentable objects of $\mathscr{K}$.
$\Phi : \mathscr{K} \to \mathscr{K}$ is a finitary endofunctor, where

$$\Phi \cong M_\Phi \otimes -$$

for a flat module

$$M : \mathscr{A}^{op} \times \mathscr{A} \longrightarrow \mathsf{Set}$$

Also,

$$U : \mathrm{Coalg}(\Phi) \longrightarrow \mathscr{K}$$

denotes the forgetful functor, where $\mathrm{Coalg}(\Phi)$ is the category of coalgebras for the endofunctor Φ.

Definition 2.14. Given a (flat) module M and an object $K \in \mathscr{K}$, the category

$$\mathsf{Complex}^K(M)$$

of *M-complexes over K and their morphisms* is defined as follows:

(1) Objects, called *M-complexes pointed in K*, are countable chains of the form

$$\cdots \xrightarrow{m_3} (a_2, x_2) \xrightarrow{m_2} (a_1, x_1) \xrightarrow{m_1} (a_0, x_0)$$

where a_i are in $\mathscr{A}$ and $x_i \in K(a_i)$, $m_i \in M(a_{i+1}, a_i)$.
Equivalently $x_i : \mathscr{A}(a_i, -) \to K$.
A single complex, for a fixed object K, as above will be denoted by $(a_\bullet, m_\bullet)^K$ for short and called K-pointed or simply pointed complexes if the object K is clear from the context.

(2) *Morphisms from* $(a_\bullet, m_\bullet)^K$ *to* $(a'_\bullet, m'_\bullet)^K$ are sequences $f_n : a_n \longrightarrow a'_n$, denoted by $(f_\bullet)$, such that all squares in the following diagram

$$\begin{array}{ccccccc} \cdots \xrightarrow{m_3} & (a_2, x_2) & \xrightarrow{m_2} & (a_1, x_1) & \xrightarrow{m_1} & (a_0, x_0) \\ & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ \cdots \xrightarrow[m'_3]{} & (a'_2, x'_2) & \xrightarrow[m'_2]{} & (a'_1, x'_1) & \xrightarrow[m'_1]{} & (a'_0, x'_0) \end{array}$$

commute and also $Kf_i(x_i) = x'_i$ hold for each $i \geq 0$.

3. Construction of the right adjoint

Our purpose is, for a given object K in $\mathscr{K}$, to construct a *cofree* Φ*-coalgebra* on K. This means the following:

Definition 3.1. A Φ-coalgebra $\kappa : \widehat{K} \longrightarrow \Phi(\widehat{K})$ together with a morphism $\varepsilon^K : \widehat{K} \longrightarrow K$ is called a *cofree* Φ*-coalgebra on* K if for any Φ-coalgebra $e : X \longrightarrow \Phi(X)$ and any morphism $\delta : X \longrightarrow K$, there exist a unique coalgebra homomorphism

$$\begin{array}{ccc} X & \xrightarrow{e} & \Phi(X) \\ {\scriptstyle e^\sharp}\downarrow & & \downarrow{\scriptstyle \Phi(e^\sharp)} \\ \widehat{K} & \xrightarrow[\kappa]{} & \Phi(\widehat{K}) \end{array}$$

such that the equality

$$\delta = \varepsilon^K \cdot e^\sharp$$

holds.

Theorem 3.2. *A cofree coalgebra for* $M \otimes -$ *on* K *exists provided the category* $\mathsf{Complex}^K(M)$ *is cofiltered.*

Proof. We are going to construct the cofree coalgebra in several steps.

Step 1: Definition of the coalgebra $(\widehat{K}, \kappa)$. Define $\widehat{K} : \mathscr{A} \longrightarrow \mathsf{Set}$ as the colimit of

$$\Big(\mathsf{Complex}^K(M)\Big)^{op} \xrightarrow{\mathrm{pr}_0^{op}} \mathscr{A}^{op} \xrightarrow{Y} [\mathscr{A}, \mathsf{Set}] \tag{3.1}$$

By our assumption, the functor $\hat{K} : \mathscr{A} \longrightarrow \mathsf{Set}$ is *flat*, being a filtered colimit of representables.

An element of $\widehat{K}(a)$ is an element of a colimit $\mathscr{A}(a_0^i, -)$, evaluated at a, where a_0^i runs over all heads of complexes in $\mathsf{Complex}^K(M)$. Thus it is an equivalence class $[f : a_0^i \longrightarrow a]$, for some i, where

$$[f : a_0^i \longrightarrow a] = [g : a_0^j \longrightarrow a]$$

if there are complex morphisms

$$\begin{array}{ccccc} \cdots \longrightarrow & (a_1^i, x_1^i) & \xrightarrow{m_1^i} & (a_0^i, x_0^i) & \\ & \nearrow{\scriptstyle f_1} & & \nearrow{\scriptstyle f_0} & \\ \cdots \longrightarrow (a_1^k, x_1^k) & \xrightarrow{m_1^k} & (a_0^k, x_0^k) & & \\ & \searrow{\scriptstyle g_1} & & \searrow{\scriptstyle g_0} & \\ \cdots \longrightarrow & (a_1^j, x_1^j) & \xrightarrow{m_1^j} & (a_0^j, x_0^j) & \end{array}$$

such that the following diagram,

$$\begin{array}{ccccc} & & a_0^i & & \\ & {\scriptstyle f_0}\nearrow & & \searrow{\scriptstyle f} & \\ a_0^k & & & & a \\ & {\scriptstyle g_0}\searrow & & \nearrow{\scriptstyle g} & \\ & & a_0^j & & \end{array}$$

commutes.

To stress the dependence on the complex, we represent an element of $\widehat{K}(a)$ as

$$[\ \cdots \overset{m_3}{\nrightarrow} (a_2,x_2) \overset{m_2}{\nrightarrow} (a_1,x_1) \overset{m_1}{\nrightarrow} (a_0,x_0) \xrightarrow{f} a\ ]$$

This equals to

$$[\ \cdots \overset{m_3}{\nrightarrow} (a_2,x_2) \overset{m_2}{\nrightarrow} (a_1,x_1) \overset{f@m_1}{\nrightarrow} (a,Kf(x_0))\ ]$$

since there is always a morphism

$$\begin{array}{ccccccc} \cdots \overset{m_3}{\nrightarrow} & (a_2,x_2) & \overset{m_2}{\nrightarrow} & (a_1,x_1) & \overset{f@m_1}{\nrightarrow} & (a,Kf(x_0)) \\ & {\scriptstyle id}\uparrow & & {\scriptstyle id}\uparrow & & {\scriptstyle f}\uparrow \\ \cdots \overset{m_3}{\nrightarrow} & (a_2,x_2) & \overset{m_2}{\nrightarrow} & (a_1,x_1) & \overset{m_1}{\nrightarrow} & (a_0,x_0) \end{array}$$

in $\mathsf{Complex}^K(M)$

The action of $\widehat{K}$ on $h : a \longrightarrow a'$ is given by postcomposition:

$$\widehat{K}(h) : \widehat{K}(a) \longrightarrow \widehat{K}(a')$$

It sends the equivalence class

$$[\ \cdots \overset{m_3}{\nrightarrow} (a_2,x_2) \overset{m_2}{\nrightarrow} (a_1,x_1) \overset{m_1}{\nrightarrow} (a_0,x_0) \xrightarrow{f} a\ ] \in \widehat{K}(a)$$

to the equivalence class

$$[\ \cdots \overset{m_3}{\nrightarrow} (a_2,x_2) \overset{m_2}{\nrightarrow} (a_1,x_1) \overset{m_1}{\nrightarrow} (a_0,x_0) \xrightarrow{h\circ f} a'\ ] \in \widehat{K}(a')$$

► $\widehat{K}$ *is the carrier of a* Φ*-coalgebra structure* $\kappa : \widehat{K} \longrightarrow M \otimes \widehat{K}$. We define the coalgebra structure κ objectwise. For each $a \in \mathscr{A}$

$$\kappa_a : \widehat{K}(a) \longrightarrow (M \otimes \widehat{K})(a) = \int^{a'} M(a',a) \times \widehat{K}(a')$$

is a map sending the equivalence class

$$[\ \cdots \overset{m_3}{\nrightarrow} (a_2,x_2) \overset{m_2}{\nrightarrow} (a_1,x_1) \overset{m_1}{\nrightarrow} (a_0,x_0) \xrightarrow{f} a\ ]$$

to the element

$$[\ a_1 \overset{m_1}{\nrightarrow} a_0 \xrightarrow{f} a\ , [\ \cdots \overset{m_2}{\nrightarrow} (a_1,x_1) \xrightarrow{id} a_1\ ]]$$

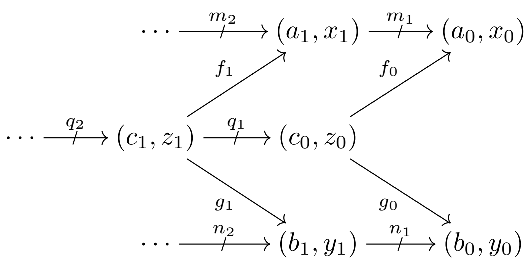
of the coend.

► κ_a *is well-defined:* Suppose the equation

$$[\ \cdots \overset{m_2}{\nrightarrow} (a_1,x_1) \overset{m_1}{\nrightarrow} (a_0,x_0) \xrightarrow{f} a\ ] = [\ \cdots \overset{n_2}{\nrightarrow} (b_1,y_1) \overset{n_1}{\nrightarrow} (b_0,y_0) \xrightarrow{g} a\ ]$$

holds.

Then there are complex morphisms



such that the following diagram,

$$\begin{array}{ccccc} & & a_0 & & \\ & \overset{f_0}{\nearrow} & & \overset{f}{\searrow} & \\ c_0 & & & & a \\ & \underset{g_0}{\searrow} & & \underset{g}{\nearrow} & \\ & & b_0 & & \end{array}$$

commutes.

Hence the equalities

$$\kappa_a([\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\,]) = \left[a_1 \overset{m_1}{\nrightarrow} a_0 \xrightarrow{f} a \;, [\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \xrightarrow{\mathrm{id}} a_1\,] \right]$$

$$\kappa_a([\,\cdots \overset{n_2}{\nrightarrow} (b_1, y_1) \overset{n_1}{\nrightarrow} (b_0, y_0) \xrightarrow{g} a\,]) = \left[b_1 \overset{n_1}{\nrightarrow} b_0 \xrightarrow{g} a \;, [\,\cdots \overset{n_2}{\nrightarrow} (b_1, y_1) \xrightarrow{\mathrm{id}} b_1\,] \right]$$

hold.

The latter two elements of the coend are identified by virtue of the equations

$$\widehat{K} f_1([\,\cdots \overset{q_2}{\nrightarrow} (c_1, z_1)\,]) = [\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \xrightarrow{\mathrm{id}} a_1\,]$$

$$\widehat{K} g_1([\,\cdots \overset{q_2}{\nrightarrow} (c_1, z_1)\,]) = [\,\cdots \overset{n_2}{\nrightarrow} (b_1, y_1) \xrightarrow{\mathrm{id}} b_1\,]$$

and

$$f@m_1@f_1 = (f \cdot f_0)@q_1 = (g \cdot g_0)@q_1 = g@n_1@g_1$$

▶ κ *is natural:* Choose any $h : a \longrightarrow b$ and consider the following diagram:

$$\begin{array}{ccc} \hat{K}(a) & \xrightarrow{\kappa_a} & \int^{a'} M(a', a) \times \hat{K}(a') \\ {\scriptstyle \widehat{K}(h)} \downarrow & & \downarrow {\scriptstyle \int^{a'} M(a', h) \times \widehat{K}(a')} \\ \widehat{K}(b) & \xrightarrow[\kappa_b]{} & \int^{a'} M(a', b) \times \widehat{K}(a') \end{array}$$

In order to prove the commutativity of the above diagram we choose an element

$$[\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\,]$$

in $\widehat{K}(a)$. Observe that κ_a sends

$$[\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\,]$$

to

$$\left[a_1 \overset{m_1}{\nrightarrow} a_0 \xrightarrow{f} a \;, [\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \xrightarrow{id} a_1\,] \right]$$

and then $\int^{a'} M(a', h) \times \widehat{K}(a')$ sends it to

$$\left[a_1 \overset{m_1}{\nrightarrow} a_0 \xrightarrow{h \cdot f} b \;, [\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \xrightarrow{id} a_1\,] \right]$$

On the other hand, $\widehat{K}(h)$ sends

$$[\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\,]$$

to

$$[\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a \xrightarrow{h} b\,]$$

that is mapped by κ_b to

$$\left[a_1 \overset{m_1}{\nrightarrow} a_0 \xrightarrow{h \cdot f} b \;, [\,\cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \xrightarrow{id} a_1\,] \right]$$

Therefore κ is natural and this completes the definition of the coalgebra $\kappa : \widehat{K} \longrightarrow M \otimes \widehat{K}$.

Step 2: Definition of the morphism $\varepsilon^K : \widehat{K} \longrightarrow K$. As before, we define ε^K objectwise. For a in $\mathscr{A}$, let

$$\varepsilon^K_a : \widehat{K}(a) \longrightarrow K(a)$$

map the element

$$[\ \cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\ ]$$

to

$$Kf(x_0)$$

▶ ε^K_a *is well-defined.* The equality

$$[\ \cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\ ] = [\ \cdots \overset{n_2}{\nrightarrow} (b_1, y_1) \overset{n_1}{\nrightarrow} (b_0, y_0) \xrightarrow{g} a\ ]$$

gives the existence of the diagram

$$\begin{array}{ccccccc}
 & & \cdots & \overset{m_2}{\nrightarrow} & (a_1, x_1) & \overset{m_1}{\nrightarrow} & (a_0, x_0) \\
 & & & {\scriptstyle f_1}\nearrow & & {\scriptstyle f_0}\nearrow & \\
\cdots & \overset{q_2}{\nrightarrow} & (c_1, z_1) & \overset{q_1}{\nrightarrow} & (c_0, z_0) & & \\
 & & & {\scriptstyle g_1}\searrow & & {\scriptstyle g_0}\searrow & \\
 & & \cdots & \overset{n_2}{\nrightarrow} & (b_1, y_1) & \overset{n_1}{\nrightarrow} & (b_0, y_0)
\end{array}$$

such that the following diagram

$$\begin{array}{ccccc}
 & & a_0 & & \\
 & {\scriptstyle f_0}\nearrow & & \searrow{\scriptstyle f} & \\
c_0 & & & & a \\
 & {\scriptstyle g_0}\searrow & & \nearrow{\scriptstyle g} & \\
 & & b_0 & &
\end{array}$$

commutes. Therefore the equality

$$Kf(x_0) = Kf(Kf_0(z_0)) = Kg(Kg_0(z_0)) = Kg(y_0)$$

holds.

▶ ε^K *is natural.* Consider $h : a \longrightarrow b$ and the diagram

$$\begin{array}{ccc}
\widehat{K}(a) & \xrightarrow{\varepsilon_a} & K(a) \\
{\scriptstyle \widehat{K}(h)}\downarrow & & \downarrow{\scriptstyle K(h)} \\
\widehat{K}(b) & \xrightarrow[\varepsilon_b]{} & K(b)
\end{array}$$

It is easy to see that both passages send

$$[\ \cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\ ] \in \hat{K}(a)$$

to the same element

$$K(h \cdot f)(x_0) \in K(b)$$

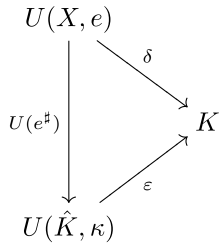
Step 3: The morphism $\varepsilon : \widehat{K} \longrightarrow K$ is U-couniversal. Recall that $U : \mathsf{Coalg}(\Phi) \longrightarrow \mathscr{K}$ denotes the forgetful functor. Hence we need to prove that for a given morphism

$$\delta : U(X, e) \longrightarrow K$$

there is a *unique* coalgebra homomorphism

$$e^\sharp : (X, e) \longrightarrow (\widehat{K}, \kappa)$$

such that the following diagram



(3.2)

commutes.

► *Existence of* $e^\sharp$. Given a coalgebra $e : X \longrightarrow M \otimes X$, where X is a flat functor, we define for each a in $\mathscr{A}$ the map

$$e^\sharp_a : X(a) \longrightarrow \widehat{K}(a)$$

in the following way.

Since X is flat, there exists an element $x \in X(a)$. By applying $e_a : X(a) \longrightarrow (M \otimes X)(a)$ we obtain an element

$$e_a(x) = [\ a_1 \overset{m_1}{\nrightarrow} a\ , x_1 \in X(a_1)]$$

Repeating the same procedure to $x_1 \in X(a_1)$ we obtain

$$e_{a_1}(x_1) = [\ a_2 \overset{m_2}{\nrightarrow} a_1\ , x_2 \in X(a_2)]$$

In this manner we construct a complex

$$\cdots \overset{m_3}{\nrightarrow} a_2 \overset{m_2}{\nrightarrow} a_1 \overset{m_1}{\nrightarrow} a$$

together with a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in X(a_n)$. Such a complex is called an *e-resolution* of $x \in X(a)$ and a choice of such will be denoted by

$$\mathsf{res}^K(x)$$

Since $\delta : X \longrightarrow K$ is a natural transformation we can take, for each $x_i \in X(a_i)$, $i \geq 0$, a sequence of elements

$$\delta_{a_1}(x_1) \in K(a_1)\ ,\ \delta_{a_2}(x_2) \in K(a_2)\ ,\ \delta_{a_3}(x_3) \in K(a_3)\ ,\ \ldots$$

In this sense we construct a *resolution over* K and define,

$$e^\sharp_a(x) = [\mathsf{res}^K(x)] = [\ \cdots \overset{m_3}{\nrightarrow} (a_2, \delta_{a_2}(x_2)) \overset{m_2}{\nrightarrow} (a_1, \delta_{a_1}(x_1)) \overset{m_1}{\nrightarrow} (a, \delta_a(x)) \xrightarrow{\text{id}} a\ ] \tag{3.3}$$

We have to verify that this definition is independent from the choice of the resolution. This is a direct application of Lemma 5.9 of [Le$_1$]. We carry out the argument in every detail for the sake of completeness.

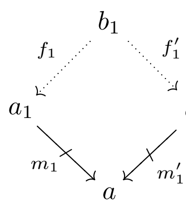
Suppose that we have two resolutions of x:

$$\begin{aligned} res^K_1(x) &= \cdots \overset{m_3}{\nrightarrow} (a_2, x_2) \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a, x) \\ res^K_2(x) &= \cdots \overset{m_3'}{\nrightarrow} (a_2', x_2') \overset{m_2'}{\nrightarrow} (a_1', x_1') \overset{m_1'}{\nrightarrow} (a, x) \end{aligned} \tag{3.4}$$

In particular, this means that the equality

$$[\ a_1 \overset{m_1}{\nrightarrow} a\ , x_1 \in X(a_1)] = [\ a_1' \overset{m_1'}{\nrightarrow} a\ , x_1' \in X(a_1')]$$

holds and therefore, by Lemma 3.2 of [Le$_1$], there exist a commutative square



(3.5)

and an element $y_1 \in X(b_1)$ such that $Xf_1(y_1) = x_1$ and $Xf_1'(y_1) = x_1'$.

In our setting (3.4) and (3.5) take the following form:

$$\cdots \xrightarrow{m_3} (a_2, \delta_{a_2}(x_2)) \xrightarrow{m_2} (a_1, \delta_{a_1}(x_1)) \xrightarrow{m_1} (a, \delta_a(x)) \tag{3.6}$$

$$\cdots \xrightarrow{m_3'} (a_2', \delta_{a_2'}(x_2')) \xrightarrow{m_2'} (a_1', \delta_{a_1'}(x_1')) \xrightarrow{m_1'} (a, \delta_a(x))$$

and

$$\begin{array}{ccccc} & & (b_1, \delta_{b_1}(y_1)) & & \\ & \swarrow^{f_1} & & \searrow^{f_1'} & \\ (a_1, \delta_{a_1}(x_1)) & & & & (a_1', \delta_{a_1'}(x_1') \\ & \searrow_{m_1} & & \swarrow_{m_1'} & \\ & & (a, \delta_a(x)) & & \end{array} \tag{3.7}$$

with the property, $Kf_1(\delta_{b_1}(y_1)) = \delta_{a_1}(x_1)$ and $Kf_1'(\delta_{b_1}(y_1)) = \delta_{a_1'}(x_1')$.

The latter comes from naturality of δ and from the equations $Xf_1(y_1) = x_1$ and $Xf_1'(y_1) = x_1'$. Indeed, for example, from the commutative square

$$\begin{array}{ccc} X(b_1) & \xrightarrow{\delta_{b_1}} & K(b_1) \\ {\scriptstyle Xf_1}\downarrow & & \downarrow{\scriptstyle Kf_1} \\ X(a_1) & \xrightarrow[\delta_{a_1}]{} & K(a_1) \end{array}$$

for $y_1 \in X(b_1)$, we obtain

$$Kf_1(\delta_{b_1}(y_1)) = \delta_{a_1}(Xf_1(y_1)) = \delta_{a_1}(x_1)$$

Our aim is to prove that the equivalence classes

$$[res_1^K(x)] = [\ \cdots \xrightarrow{m_3} (a_2, \delta_{a_2}(x_2)) \xrightarrow{m_2} (a_1, \delta_{a_1}(x_1)) \xrightarrow{m_1} (a, \delta_a(x)) \xrightarrow{id} a\ ]$$

$$[res_2^K(x)] = [\ \cdots \xrightarrow{m_3'} (a_2', \delta_{a_2'}(x_2')) \xrightarrow{m_2'} (a_1', \delta_{a_1'}(x_1')) \xrightarrow{m_1'} (a, \delta_a(x)) \xrightarrow{id} a\ ]$$

are equal.

To this end, we will construct, by induction, a commutative diagram

$$\begin{array}{ccccccccc} \cdots \xrightarrow{m_4} & (a_3, \delta_{a_3}(x_3)) & \xrightarrow{m_3} & (a_2, \delta_{a_2}(x_2)) & \xrightarrow{m_2} & (a_1, \delta_{a_1}(x_1)) & & & \\ & & & \nearrow^{f_2} & & \nearrow^{f_1} & & \searrow^{m_1} & \\ \cdots \xrightarrow{l_3} & (b_2, \delta_{b_2}(y_2)) & \xrightarrow{l_2} & (b_1, \delta_{b_1}(y_1)) & & \xrightarrow{m_1 @ f_1 = m_1' @ f_1'} & & & (a, \delta_a(x)) \\ & & & \searrow_{f_2'} & & \searrow_{f_1'} & & \nearrow_{m_1'} & \\ \cdots \xrightarrow{m_4'} & (a_3', \delta_{a_3'}(x_3')) & \xrightarrow{m_3'} & (a_2', \delta_{a_2'}(x_2')) & \xrightarrow{m_2'} & (a_1', \delta_{a_1'}(x_1')) & & & \end{array}$$

in $\mathsf{Complex}^K(M)$.

The base step, $k = 1$, is valid from (3.7).

For the inductive step, $k = n$, suppose that b_n, f_n, f'_n and $\delta_{b_n}(y_n)$ have been constructed so that the diagram

$$\begin{array}{ccccccc} & & (a_n, \delta_{a_n}(x_n)) \xrightarrow{m_n} \cdots \xrightarrow{m_2} (a_1, \delta_{a_1}(x_1)) & & & & \\ & \nearrow^{f_n} & & \nearrow^{f_1} & & \searrow^{m_1} & \\ (b_n, \delta_{b_n}(y_n)) & \xrightarrow{l_n} \cdots \xrightarrow{l_2} & & (b_1, \delta_{b_1}(y_1)) & \xrightarrow{m_1 @ f_1 = m'_1 @ f'_1} & & (a, \delta_a(x)) \\ & \searrow_{f'_n} & & \searrow_{f'_1} & & \nearrow_{m'_1} & \\ & & (a'_n, \delta_{a'_n}(x'_n)) \xrightarrow{m'_n} \cdots \xrightarrow{m'_2} (a'_1, \delta_{a'_1}(x'_1)) & & & & \end{array} \tag{3.8}$$

commutes.

Using the fact that $e : X \longrightarrow M \otimes X$ is a coalgebra and $y_n \in X(b_n)$ we deduce that

$$e_{b_n}(y_n) = [\ c \xrightarrow{q} b_n\ , z \in X(c)]$$

and this gives an element $\delta_c(z) \in K(c)$.

Hence the diagram (3.8) becomes

$$\begin{array}{ccccccc} & & (a_{n+1}, \delta_{a_{n+1}}(x_{n+1})) \xrightarrow{m_{n+1}} (a_n, \delta_{a_n}(x_n)) \xrightarrow{m_n} \cdots \xrightarrow{m_2} (a_1, \delta_{a_1}(x_1)) & & & & \\ & \nearrow^{f_n} & & \nearrow^{f_1} & & \searrow^{m_1} & \\ (c, \delta_c(z)) \xrightarrow{q} (b_n, \delta_{b_n}(y_n)) & \xrightarrow{l_n} \cdots \xrightarrow{l_2} & & (b_1, \delta_{b_1}(y_1)) & \xrightarrow{m_1 @ f_1 = m'_1 @ f'_1} & & (a, \delta_a(x)) \\ & \searrow_{f'_n} & & \searrow_{f'_1} & & \nearrow_{m'_1} & \\ & & (a'_{n+1}, \delta_{a'_{n+1}}(x'_{n+1})) \xrightarrow{m'_{n+1}} (a'_n, \delta_{a'_n}(x'_n)) \xrightarrow{m'_n} \cdots \xrightarrow{m'_2} (a'_1, \delta_{a'_1}(x'_1)) & & & & \end{array} \tag{3.9}$$

Naturality of e

$$\begin{array}{ccc} X(b_n) & \xrightarrow{\varepsilon_{b_n}} & (M \otimes X)(b_n) \\ {\scriptstyle Xf_n}\downarrow & & \downarrow{\scriptstyle (M\otimes X)(f_n)} \\ X(a_n) & \xrightarrow[\varepsilon_{a_n}]{} & (M \otimes X)(a_n) \end{array}$$

then ensures that equalities

$$\begin{aligned} [\ a_{n+1} \xrightarrow{m_{n+1}} a_n\ , x_{n+1} \in X(a_{n+1})] = e_{a_n}(x_n) = e_{a_n}(Xf_n(y_n)) &= (M \otimes X)(f_n)(e_{b_n}(y_n) \\ &= [\ c \xrightarrow{f_n @ q} a_n\ , z \in X(c)] \end{aligned}$$

hold.

Again by Lemma 3.2 of [Le$_1$] we have the commutative diagram

$$\begin{array}{ccccc} & & d & & \\ & {\scriptstyle g}\swarrow & & \searrow{\scriptstyle h} & \\ c & & & & a_{n+1} \\ & {\scriptstyle f_n @ q}\searrow & & \swarrow{\scriptstyle m_{n+1}} & \\ & & a_n & & \end{array} \tag{3.10}$$

and an element $w \in X(d)$ such that $Xg(w) = z$ and $Xh(w) = x_{n+1}$.

As before, $\delta_d(w) \in K(d)$ and from naturality of δ the diagrams

$$\begin{array}{ccc} X(d) & \xrightarrow{\delta_d} & K(d) \\ {\scriptstyle Xh}\downarrow & & \downarrow{\scriptstyle Kh} \\ X(a_{n+1}) & \xrightarrow[\delta_{a_{n+1}}]{} & Ka_{n+1} \end{array} \qquad \begin{array}{ccc} X(d) & \xrightarrow{\delta_d} & K(d) \\ {\scriptstyle Xg}\downarrow & & \downarrow{\scriptstyle Kg} \\ X(c) & \xrightarrow[\delta_c]{} & Kc \end{array}$$

commute.

Therefore the equalities

$$Kh(\delta_d(w)) = \delta_{a_{n+1}}(Xh(w)) = \delta_{a_{n+1}}(x_{n+1})\ ,\ Kg(\delta_d(w)) = \delta_c(Xg(w)) = \delta_c(z) \tag{3.11}$$

hold.

From (3.10) and (3.11) it is easy to conclude the commutativity of:

$$
\begin{array}{ccccc}
 & & (a_{n+1}, \delta_{a_{n+1}}(x_{n+1})) & & \\
 & \overset{h}{\nearrow} & & \overset{m_{n+1}}{\searrow} & \\
(d, \delta_d(w)) & & & & (a_n, \delta_{a_n}(x_n)) \\
{\scriptstyle g}\downarrow & & & \overset{f_n}{\nearrow} & \\
(c, \delta_c(z)) & \xrightarrow[q]{} & (b_n, \delta_{b_n}(y_n)) & &
\end{array}
\tag{3.12}
$$

Applying naturality of ε^K to the morphism $f'_n : b_n \longrightarrow a'_n$ and following the same procedure we obtain the diagram

$$
\begin{array}{ccccc}
 & & (a'_{n+1}, \delta_{a'_{n+1}}(x'_{n+1})) & & \\
 & \overset{h'}{\nearrow} & & \overset{m'_{n+1}}{\searrow} & \\
(d', \delta'_d(w')) & & & & (a'_n, \delta_{a'_n}(x'_n)) \\
{\scriptstyle g'}\downarrow & & & \overset{f'_n}{\nearrow} & \\
(c, \delta_c(z)) & \xrightarrow[q]{} & (b_n, \delta_{b_n}(y_n)) & &
\end{array}
\tag{3.13}
$$

From the above, the diagram (3.9) takes the shape

$$
\begin{array}{ccccc}
 & & (a_{n+1}, \delta_{a_{n+1}}(x_{n+1})) \xrightarrow{m_{n+1}} (a_n, \delta_{a_n}(x_n)) \xrightarrow{m_n} \cdots & & \\
 & \overset{h}{\nearrow} & & \overset{f_n}{\nearrow} & \\
(d, \delta_d(w)) & & & & \\
{\scriptstyle g}\downarrow & & & & \\
(c, \delta_c(z)) & \xrightarrow{q} & (b_n, \delta_{b_n}(y_n)) & \xrightarrow{l_n} & \cdots \\
{\scriptstyle g'}\uparrow & & & & \\
(d', \delta_{d'}(w')) & & & \underset{f'_n}{\searrow} & \\
 & \underset{h'}{\searrow} & & & \\
 & & (a'_{n+1}, \delta_{a'_{n+1}}(x'_{n+1})) \xrightarrow{m'_{n+1}} (a'_n, \delta_{a'_n}(x'_n)) \xrightarrow{m'_n} \cdots & &
\end{array}
\tag{3.14}
$$

Finally, from (3.10), (3.13) the following equality

$$Xg(w) = z = Xg'(w')$$

follows.

Hence flatness of X gives us an object b_{n+1} in $\mathscr{A}$, two morphisms $k : b_{n+1} \longrightarrow d$, $k' : b_{n+1} \longrightarrow d'$ and an element $y_{n+1} \in X(b_{n+1})$ such that the following equalities

$$Xk(y_{n+1}) = w\ ,\ Xk'(y_{n+1}) = w'\ ,\ g \cdot k = g' \cdot k'$$

hold.

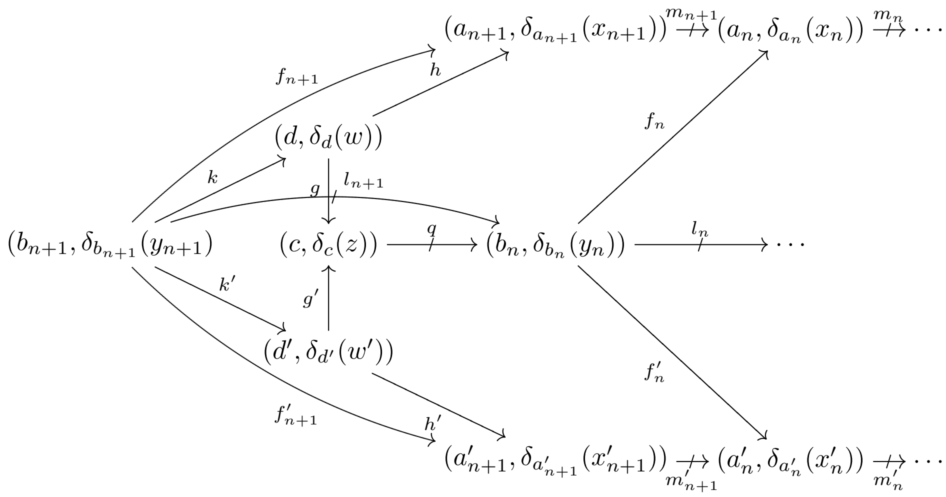
Since $\delta_{b_{n+1}}(y_{n+1}) \in Kb_{n+1}$, naturality of δ

$$
\begin{array}{ccc}
X(b_{n+1}) & \xrightarrow{\delta_{b_{n+1}}} & K(b_{n+1}) \\
{\scriptstyle X(k)}\downarrow & & \downarrow{\scriptstyle K(k)} \\
X(d) & \xrightarrow[\delta_d]{} & K(d)
\end{array}
\qquad
\begin{array}{ccc}
X(b_{n+1}) & \xrightarrow{\delta_{b_{n+1}}} & K(b_{n+1}) \\
{\scriptstyle X(k')}\downarrow & & \downarrow{\scriptstyle K(k')} \\
X(d') & \xrightarrow[\delta_{d'}]{} & K(d')
\end{array}
$$

yields

$$K(k)(\delta_{b_{n+1}}(y_{n+1})) = \delta_d(Xk(y_{n+1})) = \delta_d(w)\ ,\ K(k')(\delta_{b_{n+1}}(y_{n+1})) = \delta_{d'}(Xk'(y_{n+1})) = \delta_{d'}(w')$$

Therefore we can form the following diagram



(3.15)

and defining: $f_{n+1} := h \cdot k$, $f'_{n+1} := h' \cdot k'$, $l_{n+1} = q@(g \cdot k)$ completes the induction. Hence we proved that $e^\sharp$ is well-defined.

► *Naturality of* $e^\sharp$. To prove naturality of $e^\sharp$, choose $h : a \longrightarrow b$ and consider the following square:

$$\begin{array}{ccc} X(a) & \xrightarrow{e^\sharp_a} & \hat{K}(a) \\ {\scriptstyle X(h)}\downarrow & & \downarrow{\scriptstyle \hat{K}(h)} \\ X(b) & \xrightarrow[e^\sharp_b]{} & \hat{K}(b) \end{array}$$

Let $x \in X(a)$. Then upper road yields:

$$\begin{array}{cll} x & \xmapsto{e^\sharp_a} & [\ \cdots \overset{m_2}{\nrightarrow} (a_1, \delta_{a_1}(x_1)) \overset{m_1}{\nrightarrow} (a, \delta_a(x)) \xrightarrow{id} a\ ] \\ & \xmapsto{\hat{K}(h)} & [\ \cdots \xrightarrow{m_2} (a_1, \delta_{a_1}(x_1)) \xrightarrow{m_1} (a, \delta_a(x)) \xrightarrow{\quad h \quad} b\ ] \\ & = & [\ \cdots \xrightarrow{m_2} (a_1, \delta_{a_1}(x_1)) \xrightarrow{h@m_1} (b, Kh(\delta_a(x)))\ ] \end{array}$$

whereas the lower road gives:

$$\begin{array}{cll} x & \xmapsto{X(h)} & Xh(x) \\ & \xmapsto{e^\sharp_b} & [\ \cdots \overset{n_2}{\nrightarrow} (b_1, \delta_{b_1}(y_1)) \overset{n_1}{\nrightarrow} (b, \delta_b(Xh(x))) \xrightarrow{\text{id}} b\ ] \end{array}$$

Observe first that the equality

$$\delta_b(Xh(x)) = Kh(\delta_a(x))$$

holds by naturality of δ. Observe furhter that from the commutativity of the diagram

$$\begin{array}{ccc} X(a) & \xrightarrow{e_a} & (M \otimes X)(a) \\ {\scriptstyle X(h)}\downarrow & & \downarrow{\scriptstyle (M\otimes X)(h)} \\ X(b) & \xrightarrow[e_b]{} & (M \otimes X)(b) \end{array}$$

it follows that

$$e_b(Xh(x)) = [\ a_1 \xrightarrow{h@m_1} b\ , x_1 \in X(a_1)]$$

Consequently, we have the equality

$$[\ \cdots \overset{m_2}{\nrightarrow} (a_1, \delta_{a_1}(x_1)) \overset{h@m_1}{\nrightarrow} (b, Kh(\delta_a(x)))\ ] = [\ \cdots \overset{n_2}{\nrightarrow} (b_1, \delta_{b_1}(y_1)) \overset{n_1}{\nrightarrow} (b, \delta_b(Xh(x))) \xrightarrow{id} b\ ]$$

proving that $e^\sharp$ is natural.

▶ $e^\sharp$ *is a coalgebra homomorphism.* We need to prove that the square

$$\begin{array}{ccc} X(a) & \xrightarrow{e_a} & (M \otimes X)(a) \\ {\scriptstyle e^\sharp}\downarrow & & \downarrow{\scriptstyle M\otimes e^\sharp} \\ \hat{K}(a) & \xrightarrow[\kappa_a]{} & (M \otimes \hat{K})(a) \end{array}$$

commutes.

Indeed, for each a in $\mathscr{A}$ and from flatness of X, there is $x \in X(a)$ such that

$$\begin{aligned} (M \otimes e^\sharp)_a(e_a(x)) &= (M \otimes e^\sharp)_a([\ a_1 \overset{m_1}{\nrightarrow} a\ , x_1 \in X(a_1)]) \\ &= [\ a_1 \overset{m_1}{\nrightarrow} a\ , [\ \cdots \overset{m_2}{\nrightarrow} (a_1, \delta_{a_1}(x_1)) \xrightarrow{id} a_1\ ]] \end{aligned}$$

and

$$\begin{aligned} \kappa_a(e^\sharp_a(x)) &= \kappa_a([res^K(x)]) \\ &= [\ a_1 \overset{m_1}{\nrightarrow} a\ , [\ \cdots \overset{m_2}{\nrightarrow} (a_1, \delta_{a_1}(x_1)) \xrightarrow{id} a_1\ ]] \end{aligned}$$

by choosing the resolution of x that starts with m_1, x_1, i.e, $\mathsf{res}(x) = (\ \cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a, x)\ )$. We have proved that $e^\sharp$ is a coalgebra homomorphism.

▶ *The triangle (3.2) commutes.* To prove the commutativity of the triangle consider, as before, an object a in $\mathscr{A}$ and an element $x \in X(a)$. Then the triangle

$$\begin{array}{ccc} X(a) & & \\ & \searrow^{\delta_a} & \\ {\scriptstyle e^\sharp_a}\downarrow & & K(a) \\ & \nearrow_{\varepsilon_a} & \\ \hat{K}(a) & & \end{array}$$

commutes, since

$$\varepsilon_a(e^\sharp_a(x)) = \varepsilon_a([\ \cdots \overset{m_2}{\nrightarrow} (a_1, \delta_{a_1}(x_1)) \overset{m_1}{\nrightarrow} (a, \delta_a(x)) \xrightarrow{id} a\ ]) = Kid_a(\delta_a(x)) = id_{K(a)}(\delta_a(x)) = \delta_a(x)$$

▶ $e^\sharp$ *is the unique coalgebra map that makes (3.2) commutative.* Assume that

$$\begin{array}{ccc} U(X,e) & & \\ & \searrow^{\delta} & \\ {\scriptstyle U(\rho)}\downarrow & & K \\ & \nearrow_{\varepsilon} & \\ U(\hat{K},\kappa) & & \end{array} \tag{3.16}$$

commutes. We will show that $\rho = e^\sharp$.

For $x \in X(a)$ let $\rho_a(x) \in \hat{K}(a)$ be represented as

$$\rho_a(x) = [\ \ldots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\ ] \tag{3.17}$$

where $x_i \in K(a_i)$, $i \geq 0$, while $e^\sharp_a(x)$ be defined, with the aid of e-resolutions of x as in (3.3), as

$$e^\sharp_a(x) = [\ \ldots \overset{n_3}{\nrightarrow} (b_2, \delta_{b_2}(y_2)) \overset{n_2}{\nrightarrow} (b_1, \delta_{b_1}(y_1)) \overset{n_1}{\nrightarrow} (a, \delta_a(x)) \xrightarrow{id} a\ ] \tag{3.18}$$

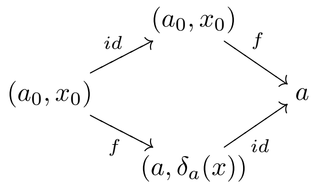
We will exhibit a mediating K-pointed complex between those appearing in (3.17) and (3.18) and appropriate K-pointed complex morphisms manifesting the equality

$$e_a^\sharp(x) = \rho_a(x)$$

The commutativity of (3.16) gives

$$Kf(x_0) = \varepsilon_a \cdot U\rho_a(x) = \delta_a(x)$$

which allows us to obtain the 0-th part of a K-pointed complex morphism



In order to proceed to depth $k > 0$ we will use the following auxiliary lemma.

Lemma 3.3. *Suppose that in the commutative square*

$$\begin{array}{ccc} X(a) & \xrightarrow{e_a} & (M\otimes X)(a) \\ {\scriptstyle \rho_a}\downarrow & & \downarrow{\scriptstyle M\otimes\rho_a} \\ \hat{K}(a) & \xrightarrow[\kappa_a]{} & (M\otimes \hat{K})(a) \end{array}$$

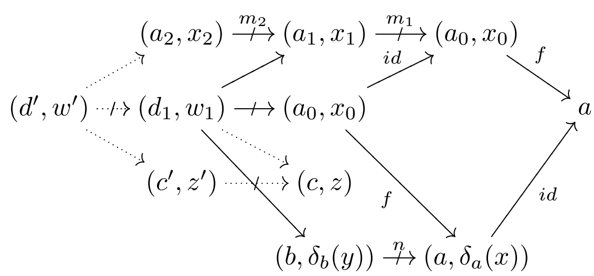
given by the fact that ρ is a morphism of coalgebras, we have

$$\rho_a(x) = [\ \dots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\ ]$$

and

$$e_a(x) = [\ b \overset{\eta}{\nrightarrow} a\ , y \in X(b)]$$

*Then there exist morphisms of depth 1 between K-pointed complexes, (*solid arrows*)*



and an extension of the diagram to another pair of morphisms of depth 1 (dotted arrows).

Remark 3.4. The latter (dotted) part will be used as a means to propagate the morphisms from depth 1 to infinity.

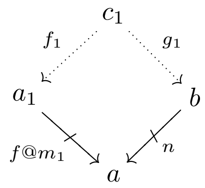
Proof of the Lemma. We have equalities

$$\begin{array}{rcl} (M\otimes\rho_a)(e_a(x)) & = & (M\otimes\rho_a)([\ b \overset{\eta}{\longrightarrow\!\!\!\!\!\!/\ \ } a\ , y \in X(b)] \\ & = & [\ b \overset{\eta}{\longrightarrow\!\!\!\!\!\!/\ \ } a\ , \rho_b(y)] \end{array}$$

and

$$\begin{array}{rcl} \kappa_a\rho_a(x) & = & \kappa_a([\ \cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{m_1}{\nrightarrow} (a_0, x_0) \xrightarrow{f} a\ ]) \\ & = & \kappa_a([\ \cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \overset{f@m_1}{\nrightarrow} (a, \delta_a(x))\ ]) \\ & = & [\ a_1 \overset{f@m_1}{\nrightarrow} a\ , [\ \cdots \overset{m_2}{\nrightarrow} (a_1, x_1) \xrightarrow{id} a_1\ ]] \end{array}$$

By Lemma 3.2 of ([Le$_1$]) there is a diagram



(3.19)

and an element

$$[\ \cdots \xrightarrow{q_2} (c_1, z_1) \xrightarrow{id} c_1\ ] \in \widehat{K}(c_1)$$

such that

$$[\ \dots \xrightarrow{q_2} (c_1, z_1) \xrightarrow{f_1} a_1\ ] = [\ \cdots \xrightarrow{m_2} (a_1, x_1)\ ] \tag{3.20}$$

and

$$[\ \cdots \xrightarrow{q_2} (c_1, z_1) \xrightarrow{g_1} b_1\ ] = \rho_b(y) \in \widehat{K}(b) \tag{3.21}$$

hold.

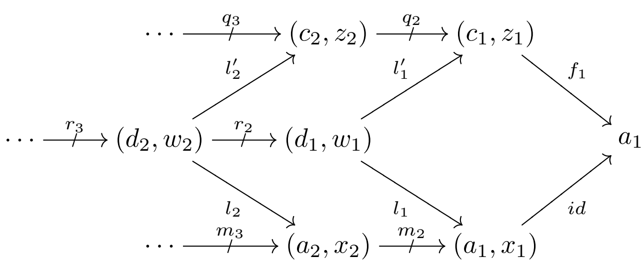
The latter element of $\widehat{K}(b)$ is some

$$[\ \cdots \xrightarrow{n_4'} (b'', y'') \xrightarrow{n_3'} (b', y') \xrightarrow{g'} b\ ] = [\ \cdots \xrightarrow{n_4'} (b'', y'') \xrightarrow{g'@n_3'} (b, Kg'(y'))\ ]$$

and the commutativity of (3.16) at b gives

$$Kg'(y') = \varepsilon_{b'} U \rho_{b'}(y') = \delta_{b'}(y')$$

On the other hand (3.20) gives a commutative diagram (involving morphisms K-pointed complexes to the left)

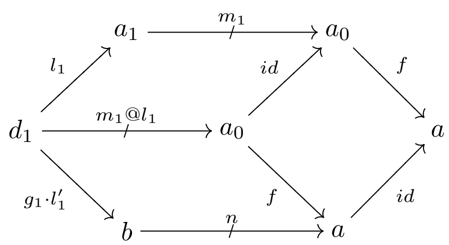


(3.22)

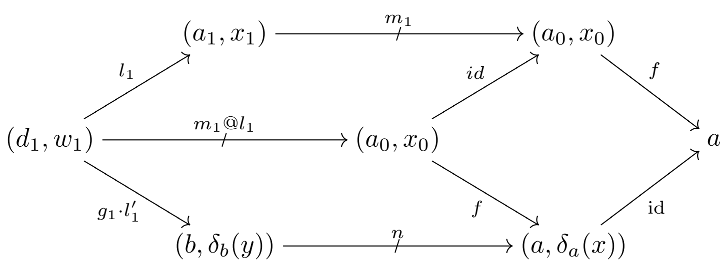
Since

$$(f@m_1) \cdot l_1 \overset{(3.22)}{=} (f@m_1) \cdot f_1 \circ l_1' \overset{(3.19)}{=} (n@g_1) \cdot l_1'$$

hold, we have a commutative diagram



hence we would have the required solid part of the diagram if



(3.23)

were a diagram of K-pointed 1-complexes.

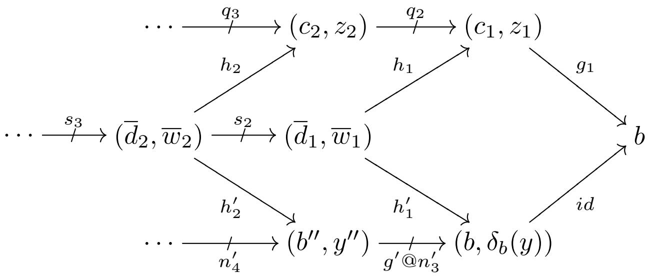
To that end we have

$$Kl_1(w_1) = x_1$$

by the lower part of (3.22), while

$$K(g_1 \cdot l_1')(w_1) = Kg_1(z_1)$$

by the upper part of (3.22). Finally, (3.21) gives



hence

$$Kg_1(z_1) = Kg_1(Kh_1(\overline{w}_1)) = Kh_1'(\overline{w}_1) = \delta_b(y)$$

showing that, indeed, we have a diagram (3.23) of K-pointed complexes of depth 1. The required dotted part in the conclusion of the Lemma comes directly from (3.22). The proof of the Lemma is finished.

□

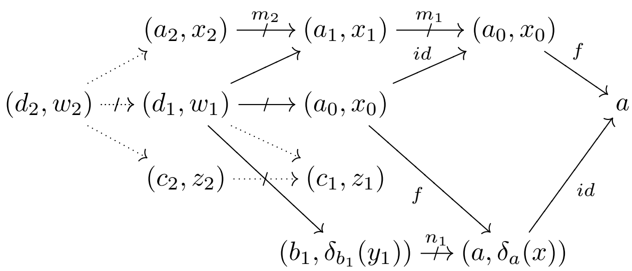
In order now to construct the desired complex mediating (3.17) and (3.18), that will eventually yield the equality $e_a^\sharp(x) = \rho_a(x)$, apply the lemma to the square

$$\begin{array}{ccc} X(a) & \xrightarrow{e_a} & (M \otimes X)(a) \\ {\scriptstyle \rho_a}\downarrow & & \downarrow{\scriptstyle M\otimes\rho_a} \\ \hat{K}(a) & \xrightarrow[\kappa_a]{} & (M \otimes \hat{K})(a) \end{array}$$

for the representation

$$e_a(x) = [\ b_1 \overset{n_1}{\nrightarrow} a\ , y_1 \in X(b_1)]$$

(beginning of a resolution of x), in order to obtain morphism of K-pointed complexes at depth 1



(3.24)

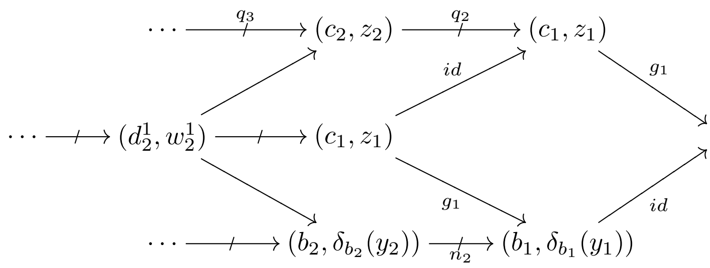
Applying the Lemma once more, for the commutative square

$$\begin{array}{ccc} X(b_1) & \xrightarrow{e_{b_1}} & (M \otimes X)(b_1) \\ {\scriptstyle \rho_{b_1}}\downarrow & & \downarrow{\scriptstyle M\otimes\rho_{b_1}} \\ \hat{K}(b_1) & \xrightarrow[\kappa_{b_1}]{} & (M \otimes \hat{K})(b_1) \end{array}$$

where

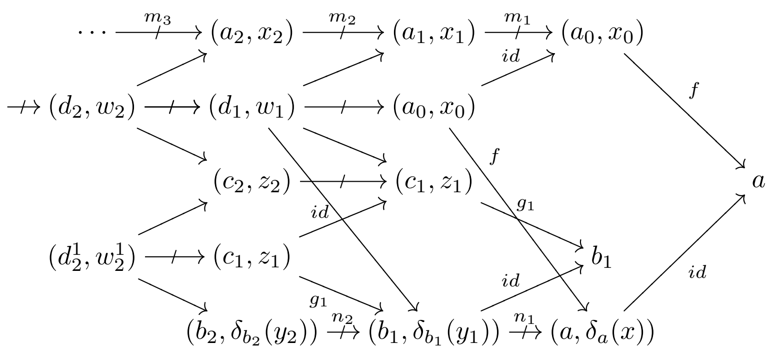
$$e_{b_1}(y_1) = [\ b_2 \overset{n_2}{\nrightarrow} b_1\ , y_2 \in X(b_2)]$$

(the continuation of a resolution of x) we get a commutative diagram



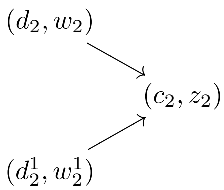
(3.25)

Inserting (3.25) into (3.24) we obtain

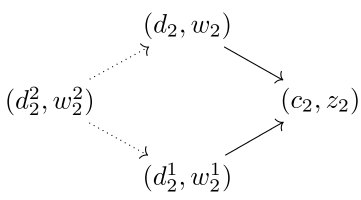


(3.26)

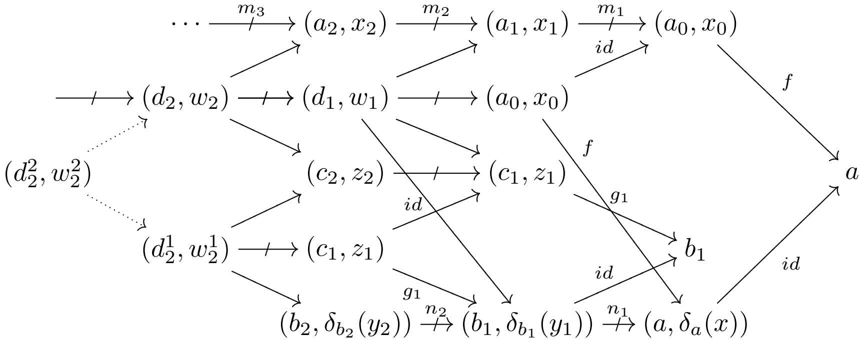
Using flatness of K we complete



into a commutative square

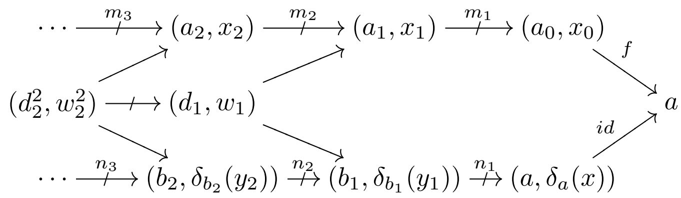


so the diagram (3.26) becomes



(3.27)

hence getting morphisms of K-pointed complexes of depth 2



and so on.

The proof of the Theorem 3.2 is finished.

□

Corollary 3.5. *If the category* $\mathsf{Complex}^K(M)$ *is cofiltered for every object* K *in* $\mathscr{K}$*, the forgetful functor* $U : \mathsf{Coalg}(\Phi) \longrightarrow \mathscr{K}$ *has a right adjoint.*

Corollary 3.6. *If* $\mathscr{K}$ *is locally finitely presentable, then the forgetful functor* $U : \mathsf{Coalg}(\Phi) \longrightarrow \mathscr{K}$ *has a right adjoint. In particular the cofree coalgebra over the final object of* $\mathscr{K}$*, i.e the final coalgebra, always exists.*

4. Cofree coalgebras over Scott-complete categories

Scott-complete categories were introduced in [A_2] as categorical generalizations of Scott-domains with a view towards denotational semantics, showing for example that they form a cartesian closed category and a limit-colimit coincidence theorem about them. They are defined as finitely accessible categories, where every diagram that has a cocone has a colimit. They have also been characterized in the same work as categories of models for theories, in a many-sorted language, given by axioms of the form

$$\forall \vec{x}(\varphi(\vec{x}) \longrightarrow \exists \vec{y}\psi(\vec{x}, \vec{y})) \tag{4.1}$$

where the existential quantifier is provably -in the theory- unique and φ, ψ are conjunctions of atomic formulae, extended by axioms of the form

$$\forall x : S(x = x \longrightarrow \bot),$$

for a specified set of sorts S (i.e axioms saying that certain sorts of variables are sent by a model to the empty set). Locally finitely presentable categories are trivial examples of Scott-complete categories since they are cocomplete (and it is well-known that they are charecterized as categories of models of axioms of the form (4.1), above). It is immediate by the above axiomatization that Scott-complete categories are closed under binary products in the category of structures. This allows us to see that there is a supply of examples of Scott-complete categories that are not l.f.p: Take a language with constants and form a theory $\mathbb{T}$ consisting of axioms of the form (4.1) and also of the form $\neg(t_i = t_j)$, for two closed terms of the theory. Then, in the category of models $\mathrm{Mod}(\mathbb{T})$, for any model M the diagram of the two projections

$$M \times M \rightrightarrows M$$

can not be coequalized. Hence, for example, partially ordered sets with distinct endpoints (and monotone maps preserving them) and commutative rings of characteristic different than any given n (and homomorphisms) form examples of Scott-complete categories. It can be shown that all Scott-complete categories can be axiomatized in this way too, but this would demand some work that falls outside the scope of this article. In particular it follows from the identification of such categories as categories of points of closed subtoposes of classifying toposes of limit theories and the fact that closed topologies correspond to subobjects of the presheaf represented by the closed terms model of such a theory.

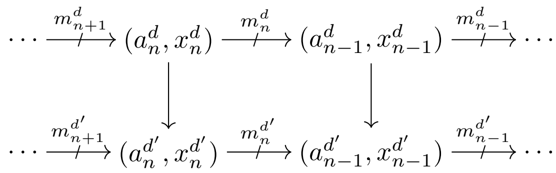
Proposition 4.1. *Suppose* $\mathscr{K}$ *is a Scott complete category and let* Φ *be a finitary endofunctor of* $\mathscr{K}$*. Then the cofree coalgebra for* Φ *exists.*

Proof. By Corollary 3.6 it suffices to show that the category $\mathsf{Complex}^K(M)$ is cofiltered for every K in $\mathscr{K}$. Let

$$D : \mathscr{D} \longrightarrow \mathsf{Complex}^K(M)$$

be a finite diagram. We will show the diagram has as a cone $(c_\bullet, z_\bullet)^K$.

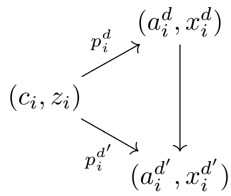
For a given $d \longrightarrow d' \in \mathscr{D}$ we have



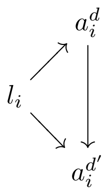
Notice that (a_i^d, x_i^d) , $(a_i^{d'}, x_i^{d'})$, for $i \geq 0$, are objects of the category $\mathsf{elts}(K)$. Since K is a flat functor for each (a_i^d, x_i^d) , $(a_i^{d'}, x_i^{d'})$ there are cospans (c_i, z_i). Hence

$$\mathscr{D} \xrightarrow{D} \mathsf{Complex}^K(M) \xrightarrow{pr_i} \mathscr{K}$$

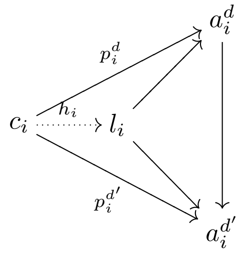
has a cone



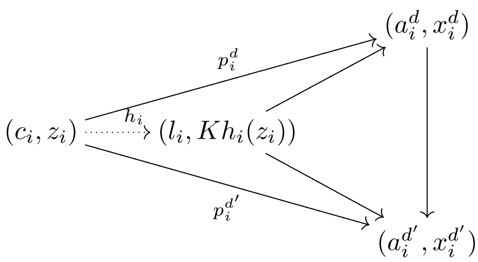
and, from the fact that $\mathscr{K}$ is Scott complete, it also has a limit



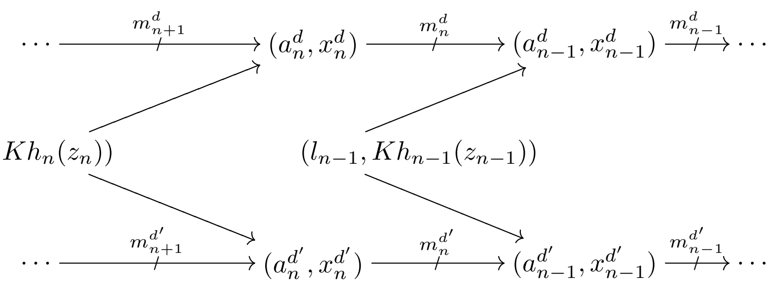
Considering the factorization $h_i : c_i \longrightarrow l_i$



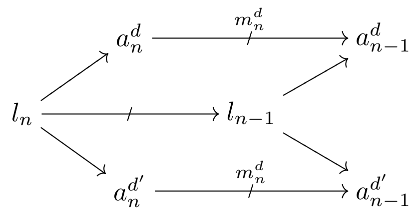
we obtain a diagram



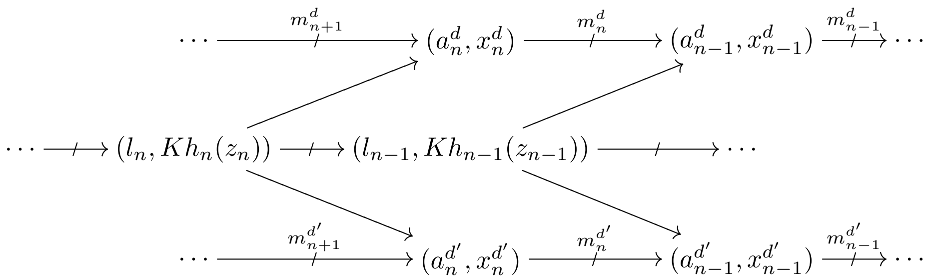
Then we take the diagram



Finally, by flatness of M, there are commutative diagrams:



hence we obtain the diagram



which is the desired cone in $\mathsf{Complex}^K(M)$. □

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