Completeness of Cocompletions

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Abstract

We study several possible weakenings of the notion of limit and the associated notions of completeness for a category. We examine the relations among the various proposed notions of weakened completeness conditions. We use these conditions in the analysis of the existence of limits inside completions of categories under colimits. We further characterize when the completion of a category under finite colimits has finite limits with the aid of a condition requiring that reflexive symmetric graphs have bounded transitive hulls.

Key words: weak limit, cocompletion

1 Introduction

It is well-known that the existence of (finite) limits in a free cocompletion of a category $\mathbb{C}$ under (finite) colimits is not automatically guaranteed. It turns out, as we show below, that existence of such limits is kind of a weak completeness condition on the category $\mathbb{C}$. Such a weak completeness condition comes as no surprise. In several similar questions a condition on a cocompletion turns out to be equivalent to some weak version of it on the original category. For example, familial completeness in a category is equivalent to finite completeness in its

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sum completion ([HT]) or extensivity of the exact completion of a category is equivalent to near extensivity of the completed category ([LV]).

To be able to state our goal more precisely, note that one can weaken the classical notion of a limit in various ways. Given a diagram $D : \mathbb{I} \longrightarrow \mathbb{C}$, a *limit of* D exists if and only if the functor

$$\mathrm{Cone}(D) : \mathbb{C}^{op} \longrightarrow \mathsf{Set}$$

is representable, where $\mathrm{Cone}(D)$ assigns a set of cones with vertex X to an object X. If we now relax the requirement of representability of $\mathrm{Cone}(D)$, we come up with weaker notions of a limit. Two such weak completeness properties have been considered extensively in the literature: The existence of a *weak limit*, which amounts to the condition that the cone functor is covered epimorphically by a representable, and the existence of a *multilimit* (or, *familial limit*), which amounts to the condition that the cone functor is isomorphic to a coproduct of representables. These two weaker limit notions have proved to be quite essential in the development of certain aspects of category theory. Their study appears to be indispensible, as it is imposed by examples such as the homotopy category of topological spaces and the category of fields, respectively.

Following this spirit, several other weakenings of the notion of limit may be proposed: One may consider the case that the cone functor, as an object in the category of Set-valued functors on $\mathbb{C}$, is finitely generated, or finitely presentable, or coherent, etc.

It is worth noticing that all the generalizations of the notion of limit proposed in the current paper are quite meaningful since they seem to arise naturally in various independent contexts. Historically, one first source of such conditions is [SGA4], Exposé VI, where they are considered in connection with the question when a presheaf topos is coherent. Next, another early reference that encounters such weakened notions of limit is [F]. There, the focus is on the distinction between small and large cone functors. Yet another context where similar conditions come up, though in their poset guise, is that of domain theory. There they appear as part of the characterizations of coherent domains (i.e., those domains whose Scott compact open subsets are closed under intersection). They are encoded in the "2/3-SFP" condition — a notion that proved to be very useful in domain theory. In fact, the attempt to generalize this latter condition to a fully categorical context lead the first-named author to consider one of the notions proposed here and to relate it to the question when the completion of a category under finite colimits has finite limits [K_1]. More recently such conditions were related to the question when the classifying topos of a geometric theory is a presheaf topos [B] and when the theory of flat functors on a category admits a coherent axiomatization [BKR].

Surprisingly, all these natural weak completeness concepts that we introduce in Section 3 below turn out to be equivalent in the following sense: If the cone functors of *all* finite diagrams in a category satisfy simultaneously either of the proposed finiteness conditions then they simultaneously satisfy any other of them. In other words, we prove that:

> There are many possible natural weakenings of the notion of a finite limit of a diagram that we propose but when it comes to completeness they are all equivalent.

In fact, we prove this result for α-small diagrams and α-smallness conditions on the cone functor, where α is a regular cardinal.

This "equivalence of limit concepts" is used further in Sections 4 and 5 to derive results on the existence of limits in colimit cocompletions. It turns out, that there is a difference of whether one considers finite diagrams or α-small ones for uncountable α:

(1) For an uncountable regular cardinal α, our main result of Section 4 is that when a category has α-small cone functors for α-small diagrams then the cocompletion of a small category under α-colimits always has finite limits. When α is inaccessible we may further deduce that the cocompletion of a small category under α-colimits has α-small limits. As a corollary of the main result of this section we get that the cocompletion of a left exact small category under α-small colimits is again left exact, when α is uncountable.

    Furthermore, in this case, a left exact functor between left exact categories induces a left exact (as well as α-colimit preserving) functor between the α-cocompletions. That is, the α-colimit cocompletion monad distributes over the finite limit completion monad on Cat. This may be a result of further interest, in view of the importance of the countable colimit completion in the study of computational nondeterminism [CPW].

(2) The existence of finite limits in free cocompletions under finite colimits is discussed in Section 5. We characterize the case when these limits exist with the aid of the weak completeness condition that we propose and an extra condition that requires the existence of bounded transitive hulls for reflexive symmetric graphs in the sum completion of the given category.

2 Several Old Concepts

In this section we recall several notions of "finiteness" of a functor $F : \mathbb{C}^{op} \longrightarrow \mathsf{Set}$ on a small category $\mathbb{C}$. We parametrize, however, these notions by a regular cardinal α, so that we will be later able to handle the situation "F is of size

$< \alpha$".

All notions introduced here are either standard (see, e.g., [AR]) or due to Peter Freyd [F], modified to incorporate α.

Definition 2.1. Let α be a regular cardinal. We say that a functor $F : \mathbb{C}^{op} \longrightarrow \mathsf{Set}$ is

(1) *α-presentable*, if it is an α-presentable object of the presheaf topos $[\mathbb{C}^{op}, \mathsf{Set}]$, i.e., if the hom-functor

$$[\mathbb{C}^{op}, \mathsf{Set}](F, -) : [\mathbb{C}^{op}, \mathsf{Set}] \longrightarrow \mathsf{Set}$$

preserves α-filtered colimits.

(2) *α-generated*, if there is a regular epimorphism

$$e : \coprod_{i \in I} \mathbb{C}(-, C_i) \longrightarrow\!\!\!\!\!\rightarrow F$$

where the set I has cardinality $< \alpha$. In this situation, e is called an *α-small cover* of F.

(3) *α-lucid*, if it is α-generated and if for every pair $u, v : P \longrightarrow F$ (P an arbitrary functor), the equalizer

$$E \longrightarrow P \underset{v}{\overset{u}{\rightrightarrows}} F$$

is an α-generated functor.

(4) *α-coherent* ([J$_1$]), if it is α-generated and if for every pair $u : S \longrightarrow F$, $v : T \longrightarrow F$ (where S and T are α-generated), the pullback

$$\begin{array}{ccc} P & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle v} \\ S & \underset{u}{\longrightarrow} & F \end{array}$$

is an α-generated functor.

Remark 2.2.

(1) Freyd speaks of *petty* functors where we use the term *generated*. His use of the notions petty and lucid is intended to capture the distinction between *small*, in the sense of having the size of a set, as opposed to *large*, in the sense of having the size of a class. We use his notions (and subsequently his results) restricted to a regular cardinal α, frequently letting $\alpha = \aleph_0$.

(2) Every representable functor $\mathbb{C}(-, C)$ is an α-presentable functor, for every α. In fact, a representable functor is *absolutely presentable*, i.e.,

$$[\mathbb{C}^{op}, \mathsf{Set}](\mathbb{C}(-, C), -) : [\mathbb{C}^{op}, \mathsf{Set}] \longrightarrow \mathsf{Set}$$

preserves *all* colimits. Moreover, every representable functor is projective with respect to regular epimorphisms.

(3) Every α-presentable functor is clearly α-generated.

(4) There is another concept of coherence in the literature. In [J$_2$], a functor F is called α-coherent, if F is α-generated and if for every arrow $u : S \longrightarrow F$ (where S is α-generated), the kernel pair

$$\begin{array}{ccc} K & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle u} \\ S & \xrightarrow[u]{} & F \end{array}$$

is an α-generated functor.

Clearly, every α-coherent functor F in the sense of Definition 2.1 is coherent in the above sense.

(5) In the definition of α-lucid we can restrict ourselves to representable functors P — this is proved in [F]. Moreover, every α-lucid functor is α-coherent in the sense of [J$_1$]. This follows from the fact that pullbacks can be constructed via products and equalizers and the fact that products of α-generated functors are themselves α-generated.

Lemma 2.3. *For $F : \mathbb{C}^{op} \longrightarrow \mathsf{Set}$, the following are equivalent:*

(1) F is α-generated.

(2) Every cover of F contains an α-small subcover, i.e., F is α-compact.

PROOF. (1) implies (2): Fix an α-small cover

$$e : \coprod_{i \in I} \mathbb{C}(-, C_i) \longrightarrow\!\!\!\!\!\rightarrow F$$

and consider any cover $f : \coprod_{j \in J} G_j \longrightarrow\!\!\!\!\!\rightarrow F$. Since every representable functor is absolutely presentable and projective, see Remark 2.2, there exists a factorization

$$\begin{array}{ccc} \coprod_{j \in J} G_j & \xrightarrow{\quad f \quad} & F \\ {\scriptstyle \mathsf{in}_{j(i)}}\uparrow & & \uparrow{\scriptstyle e} \\ G_{j(i)} & \xleftarrow{\quad e_i \quad} & \coprod_{i \in I} \mathbb{C}(-, C_i) \\ & & {\scriptstyle \mathsf{in}_i}\uparrow \\ & & \mathbb{C}(-, C_i) \end{array}$$

It suffices to prove that the induced map

$$\overline{e} : \coprod_{i \in I} G_{j(i)} \longrightarrow F$$

is an epimorphism. To that end, consider a pair u, v, such that $u \cdot \overline{u} = v \cdot \overline{v}$ holds. Then $u \cdot e = v \cdot e$ and $u = v$ follows since e is an epimorphism.

(2) implies (1): Since any $F : \mathbb{C}^{op} \longrightarrow \mathsf{Set}$ is (isomorphic to) a small colimit of representables, every F can be covered by a set of representables, say,

$$e : \coprod_{i \in I} \mathbb{C}(-, C_i) \longrightarrow\!\!\!\!\!\rightarrow F$$

whose α-small subcover

$$\coprod_{i \in J} \mathbb{C}(-, C_i) \longrightarrow\!\!\!\!\!\rightarrow F$$

proves that F is α-generated. □

3 Several New Concepts of a Limit

Given a diagram $D : \mathbb{I} \longrightarrow \mathbb{C}$, then a limit of D exists if and only if the functor

$$\mathrm{Cone}(D) : \mathbb{C}^{op} \longrightarrow \mathsf{Set}$$

is representable, where $\mathrm{Cone}(D)$ assigns a set of cones with vertex X to an object X. Of course, the representing object, say, L, of $\mathrm{Cone}(D)$, together with a natural isomorphism $\mathrm{Cone}(D) \cong \mathbb{C}(-, L)$ provides us with a limit cone for D with vertex L. If we now relax the requirement of representability of $\mathrm{Cone}(D)$, we come up with weaker notions of a limit.

Two such weak completeness properties have been considered extensively in the literature: The existence of a *weak limit* W for the diagram D, which amounts to the condition that the cone functor is covered epimorphically by a representable $\mathbb{C}(-, W) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Cone}(D)$ and the existence of a *multilimit* (or, *familial limit*) $\{P_i \mid i \in I\}$, which amounts to the condition that the cone functor is isomorphic to a coproduct of representables $\mathrm{Cone}(D) \cong \coprod_{i \in I} \mathbb{C}(-, P_i)$.

Definition 3.1. A diagram $D : \mathbb{I} \longrightarrow \mathbb{C}$ has

(1) an *αm-limit*, if $\mathrm{Cone}(D)$ is α-generated.
(2) an *α-plurilimit*, if $\mathrm{Cone}(D)$ is α-presentable.
(3) an *α-lucid-limit*, if $\mathrm{Cone}(D)$ is α-lucid.
(4) an *α-coherent-limit*, if $\mathrm{Cone}(D)$ is α-coherent.

Example 3.2. In the category FinLinOrd of finite linear orders and monotone maps preserving the end points, the pair of objects

$$A = \boxed{\bullet_0 \leq \bullet_a \leq \bullet_1} \quad \text{and} \quad B = \boxed{\bullet_0 \leq \bullet_b \leq \bullet_1}$$

does not have a coproduct, yet, the pluricoproduct exists.

Proposition 3.3. *For an α-small diagram $D : \mathbb{D} \longrightarrow \mathbb{C}$ the following conditions are equivalent:*

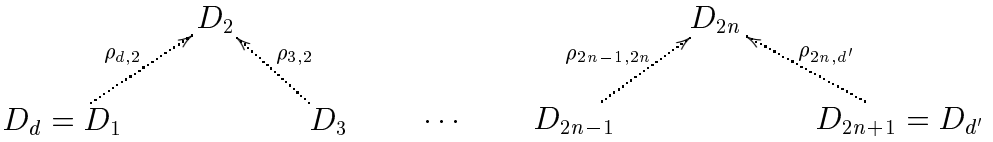
(1) D has an αm-limit.
(2) There exists an α-small set P of objects of $\mathbb{C}$ such that:
  (a) The objects in P are vertices of cones for D.
  (b) Every cone for D factors through one having a vertex in P.
(3) The category of elements of the limit for the diagram $y \cdot D$ induced by the Yoneda embedding inside $[\mathbb{C}^{op}, \mathsf{Set}]$ has an α-small weakly final set of objects $P/\lim(y \cdot D)$

We defer giving a proof for the above proposition as it will become apparent from the proof of the next proposition which elaborates on the notion of a plurilimit.

Given a diagram $D : \mathbb{D} \longrightarrow \mathbb{C}$ and cones $\langle L, \lambda : L \longrightarrow D\rangle$, $\langle L', \lambda' : L' \longrightarrow D\rangle$ for it, let us say that that a morphism $x : L \longrightarrow L'$ is a *morphism of cones* if, for every d in $\mathbb{D}$, we have

$$\lambda'_d \cdot x = \lambda_d$$

Finally, given a finite diagram $D : \mathbb{D} \longrightarrow \mathbb{C}$ we say that the arrows $x_d : A \longrightarrow D_d$, $x_{d'} : A \longrightarrow D_{d'}$ are *connected by a compatibility zig-zag* if there is a zig-zag



in the image of D joining D_d and $D_{d'}$ and arrows $\alpha_l : A \longrightarrow D_{2l+1}$ to the odd-numbered vertices of the zig-zag, such that $\rho_{d,2} \cdot x_d = \rho_{3,2} \cdot \alpha_1$, $\rho_{3,4} \cdot \alpha_1 = \rho_{5,4} \cdot \alpha_2$, $\ldots$, $\rho_{2n,d'} \cdot \alpha_n = \rho_{2n,d'} \cdot x_{d'}$.

Proposition 3.4. *For an α-small diagram $D : \mathbb{D} \longrightarrow \mathbb{C}$ the following conditions are equivalent:*

(1) D has an α-plurilimit.
(2) There exists an α-small subcategory $\mathbb{P}$ of $\mathbb{C}$ such that
  (a) The objects of $\mathbb{P}$ are vertices of cones for D and the morphisms in $\mathbb{P}$ are morphisms of these cones.
  (b) Every cone for the given diagram factors through one having a vertex in $\mathbb{P}$.
  (c) Every two factorizations of a cone through cones in $\mathbb{P}$ are connected by a compatibility zig-zag.
(3) The category of elements of the limit for the diagram $y \cdot D$ induced by the Yoneda embedding inside $[\mathbb{C}^{op}, \mathsf{Set}]$ has an α-small final subcategory $\mathbb{P}/\lim(y \cdot D)$.

PROOF. We only give a sketch of the main arguments as conditions (2) and (3) contain quite straightforward reformulations of the notion of plurilimit.

(1) $\Leftrightarrow$ (2)

(i) Every vertex of the α-plurilimit is a cone for the given diagram: Denote, for every object s in $\mathbb{P}$, by $\pi_{s,_}$ the D-cone with vertex Ps, obtained as the image of the equivalence class $[\mathrm{id} : Ps \longrightarrow Ps]$ in $\mathrm{colim}_t\, \mathbb{C}(Ps, Pt)$ under the isomorphism

$$\alpha_{Ps} : \operatorname*{colim}_t \mathbb{C}(Ps, Pt) \cong \mathrm{Cone}(D)(Ps).$$

Naturality of α gives immediately that every Pf, for $f : s \longrightarrow s'$ in $\mathbb{P}$, is a morphism of D-cones from $\pi_{s,_}$ to $\pi_{s',_}$. Consider the following diagram:

$$\begin{array}{ccccc}
[\mathrm{id} : Ps' \longrightarrow Ps'] & \in & \mathrm{colim}_t\, \mathbb{C}(Ps', Pt) \xrightarrow{\alpha_{Ps}} \mathrm{Cone}(D)(Ps') & \ni & \pi_{s',_} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{colim}_t\, \mathbb{C}(Pf,Pt)} & & \downarrow{\scriptstyle \mathrm{Cone}(D)(Pf)} \\
[Pf : Ps \longrightarrow Ps'] & \in & \mathrm{colim}_t\, \mathbb{C}(Ps, Pt) \xrightarrow[\alpha_{Ps'}]{} \mathrm{Cone}(D)(Ps) & \ni & \pi_{s',_} \cdot Pf
\end{array}$$

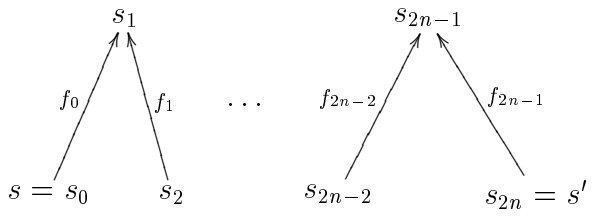
Since $[Pf : Ps \longrightarrow Ps'] = [\mathrm{id} : Ps \longrightarrow Ps] \in \mathrm{colim}_t\, \mathbb{C}(Ps, Pt)$, we conclude that $\pi_{s',_} \cdot Pf = \pi_{s,_}$ as desired.

(ii) Every cone for the diagram factors through one of the vertices of the plurilimit: Every D-cone $\gamma__$ with vertex C factors through some distinguished cone $\pi_{s,_}$ via $c : C \longrightarrow Ps$. This follows from the isomorphism

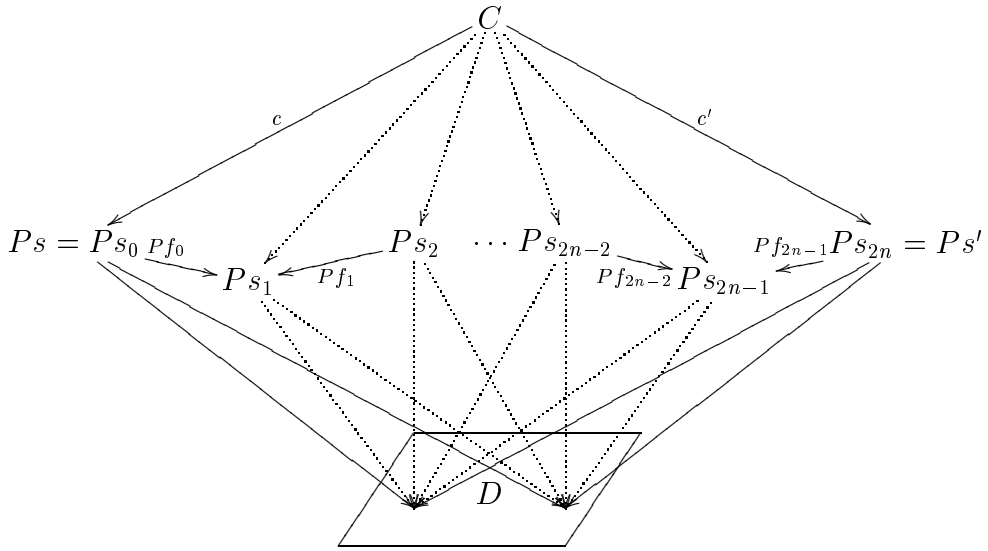
$$[c : C \longrightarrow Ps] \in \mathrm{colim}_t\, \mathbb{C}(C, Pt) \xrightarrow{\alpha_C} \mathrm{Cone}(D)(C) \ni \gamma$$

Using naturality of α in the same manner as above, we conclude that $\gamma = \pi_{s,_} \cdot c$, i.e., c is a morphism of D-cones.

(iii) Every two factorizations of a cone for the given diagram through vertices of the plurilimit are connected: If $\gamma \in \mathrm{Cone}(D)(C)$ factors through $\pi_{s,_}$ and $\pi_{s',_}$ via $c : C \longrightarrow Ps$ and $c' : C \longrightarrow Ps'$, resp., then c and c' represent the same element inside the colimit $\mathrm{colim}_s\, \mathbb{C}(C, P_s)$ thus, from the construction of colimits in Set, there exists a zig-zag



whose image under P connects c and c', i.e., there is a diagram



with the necessary commutativities holding.

(2) $\Leftrightarrow$ (3): Let $P : \mathbb{P} \longrightarrow \mathbb{C}$ be the α-plurilimit for a diagram $D : \mathbb{D} \longrightarrow \mathbb{C}$. Then the category of elements of the limit for the diagram induced by the Yoneda embedding inside $[\mathbb{C}^{op}, \mathsf{Set}]$ has an α-small final subcategory $P/\lim(y \cdot D)$: Indeed, every $x : y(A) \longrightarrow \lim(y \cdot D)$ factors through a $\pi_s : y(P_s) \longrightarrow \lim(y \cdot D)$, from condition (b). Moreover, every two such factorizations are connected by a zig-zag, from condition (c) of (2). Thus we have that for the inclusion

$$j : P/\lim(y \cdot D) \longrightarrow \mathsf{elts}(\lim(y \cdot D))$$

and every $x \in \mathsf{elts}(\lim(y \cdot D))$ the category x/j is inhabited and connected. $\square$

We show that all the above concepts concide when we restrict ourselves to *α-small diagrams* D, i.e., to $D : \mathbb{I} \longrightarrow \mathbb{C}$ where the category $\mathbb{I}$ has $< \alpha$ morphisms and where the category $\mathbb{C}$ has all such limits. (We use the obvious terminology: $\mathbb{C}$ is αm-complete, if it has αm-limits of all α-small diagrams. etc.) We use the terminology "fm-complete" instead of "$\aleph_0$m-complete".

Theorem 3.5. *For a category $\mathbb{C}$, the following are equivalent:*

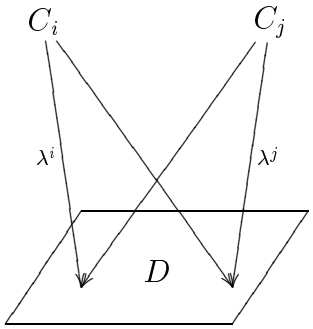
(1) $\mathbb{C}$ is αm-complete.
(2) $\mathbb{C}$ is α-pluri-complete.
(3) $\mathbb{C}$ is α-lucid-complete.
(4) $\mathbb{C}$ is α-coherent-complete.

PROOF. Implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) follow by Remark 2.2.

(1) $\Rightarrow$ (2): Take an α-small diagram $D : \mathbb{I} \longrightarrow \mathbb{C}$ and form an αm-limit

$$e : \coprod_{i \in I} \mathbb{C}(-, C_i) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Cone}(D)$$

Let $K \overset{p_1}{\underset{p_1}{\rightrightarrows}} \coprod_{i \in I} \mathbb{C}(-, C_i)$ be a kernel pair of e. We prove that K is α-generated: for every pair $(i, j) \in I \times I$ and for every pair (C_i, λ^i), (C_j, λ^j) of cones for D, consider the situation



as an α-small diagram $D^{(i,j)}$ in $\mathbb{C}$ and let

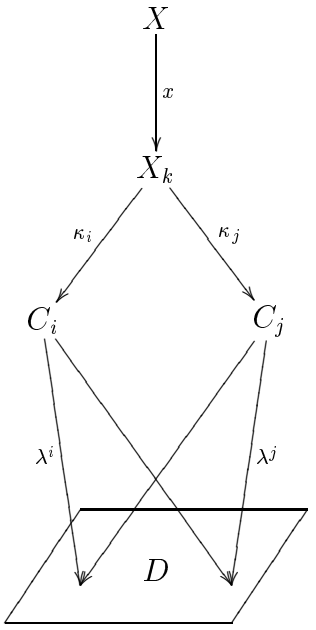
$$e^{(i,j)} : \coprod_{k \in I^{(i,j)}} \mathbb{C}(-, X_k) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Cone}(D^{(i,j)})$$

be its αm-limit in $\mathbb{C}$. Let

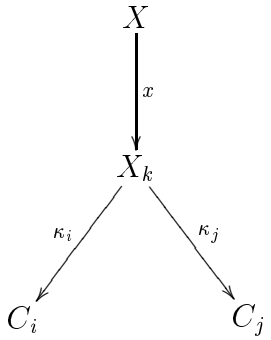
$$\varepsilon : \coprod_{(i,j) \in I \times I} \coprod_{k \in I^{(i,j)}} \mathbb{C}(-, X_k) \longrightarrow K$$

be a map that

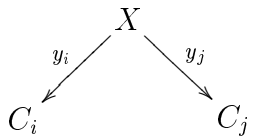
sends



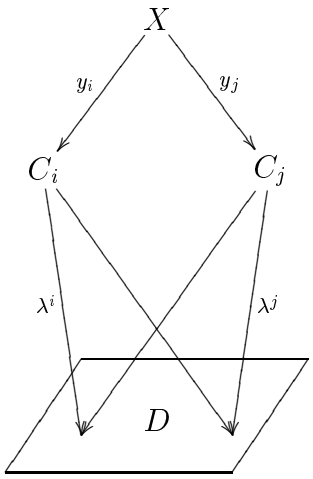
to



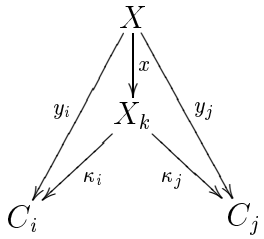
More precisely, ε sends x to the pair $(\kappa_i \cdot x, \kappa_j \cdot x)$. It suffices to show that ε is an epimorphism. Indeed, given a span



in $K(X)$, observe that



is a cone for $D^{(i,j)}$, thus it factors through an αm-limit of $D^{(i,j)}$:



This proves that K is α-generated. It now follows that the coequalizer

$$\coprod_{(i,j)\in I\times I} \coprod_{k\in I^{(i,j)}} \mathbb{C}(-, X_k) \xrightarrow{\varepsilon} K \underset{p_2}{\overset{p_1}{\rightrightarrows}} \coprod_{i\in I} \mathbb{C}(-, C_i) \xrightarrow{e} \mathrm{Cone}(D)$$

gives us an α-presentation of $\mathrm{Cone}(D)$.

(4) $\Rightarrow$ (2): This follows trivially from the fact that each α-coherent object is α-presentable.

(1) $\Rightarrow$ (3): Consider an α-small diagram $D : \mathbb{I} \longrightarrow \mathbb{C}$. We know from assumptions that $\mathrm{Cone}(D)$ is α-generated.

Any pair

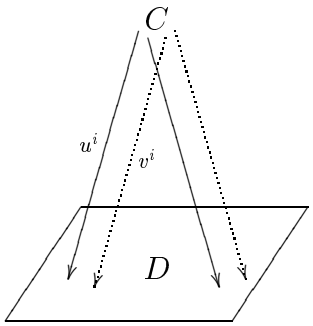
$$\mathbb{C}(-, C) \underset{v}{\overset{u}{\rightrightarrows}} \mathrm{Cone}(D)$$

amounts to a pair (C, u^i), (C, v^i) of cones for D with vertex C. To consider an equalizer

$$E \longrightarrow \mathbb{C}(-, C) \underset{v}{\overset{u}{\rightrightarrows}} \mathrm{Cone}(D)$$

at an object X is the same thing as to consider a morphism $x : X \longrightarrow C$ merging all u^i's with v^i's, i.e., it is a cone for the "double diagram"

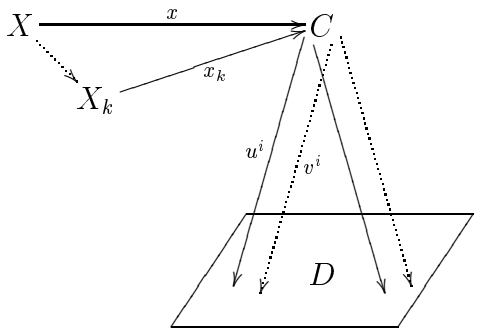
$D^* =$



Since D^* is an α-small diagram, it has an αm-limit

$$\coprod_{k \in I} \mathbb{C}(-, X_k) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Cone}(D^*)$$

hence there is an α-small family $x_k : X_k \longrightarrow C$, $k \in I$, that covers E:



□

Remark 3.6. In the presence of either of the above conditions coherence (in either sense) and lucidness mean the same thing.

4 α-small Limits in α-colex($\mathbb{C}$), $\alpha > \aleph_0$

We may detect αm-completeness of $\mathbb{C}$ by looking at either α-fam($\mathbb{C}$) (the free completion of $\mathbb{C}$ under α-small coproducts) or at α-colex($\mathbb{C}$) (the free completion of $\mathbb{C}$ under α-small colimits).

Proposition 4.1. *For a category $\mathbb{C}$, consider the following conditions:*

(1) $\mathbb{C}$ *is* α*m-complete.*
(2) α-colex($\mathbb{C}$) *has limits of* α*-small diagrams of representables.*
(3) α-fam($\mathbb{C}$) *has weak limits of* α*-small diagrams.*

Then (3) $\Rightarrow$ (1) $\Leftrightarrow$ (2). *Furthermore, when* α *is a strong limit cardinal, then* (3) $\Leftrightarrow$ (1).

PROOF. (1) $\Leftrightarrow$ (2), since, by Theorem 3.5, (1) is equivalent to α-pluricompleteness of $\mathbb{C}$ and α-plurilimits of α-small diagrams in $\mathbb{C}$ are precisely limits of α-small diagrams of representables in α-colex($\mathbb{C}$).

(3) $\Rightarrow$ (1): Let $D : \mathbb{I} \longrightarrow \mathbb{C}$ be an α-small diagram. Then $\eta \cdot D : \mathbb{I} \longrightarrow \alpha$-fam($\mathbb{C}$) has a weak limit $W = (W_j)_{j \in J}$, with the cardinality of J being less than α. Then the family $(W_j)_{j \in J}$ is an αm-limit of D.

We prove that, in case of a strong limit α, (1) $\Rightarrow$ (3): Firstly, observe that αm-limits of α-small diagrams in $\mathbb{C}$ are precisely weak limits of α-small diagrams of representables in α-fam($\mathbb{C}$).

We prove that α-fam($\mathbb{C}$) has

(a) weak products of α-small diagrams,
(b) weak equalizers,
(c) weak terminal object.

Then assertion (3) follows from (a), (b), (c) (cf. [CV], Proposition 1).

(a) Let F_i, $i \in I$, be an α-small family in α-fam($\mathbb{C}$). Then, for each $i \in I$, there is an α-small cover

$$e_i : \coprod_{j \in J_i} \mathbb{C}(-, C_j) \longrightarrow\!\!\!\!\!\rightarrow F_i$$

Consider the product

$$\prod_{i \in I} e_i : \prod_{i \in I} \coprod_{j \in J_i} \mathbb{C}(-, C_j) \longrightarrow\!\!\!\!\!\rightarrow \prod_{i \in I} F_i$$

in $[\mathbb{C}^{op}, \mathsf{Set}]$. The functor on the left is covered as follows:

$$\coprod_{f \in K} \prod_{i \in I} \mathbb{C}(-, C_{f(i)}) \longrightarrow\!\!\!\!\!\rightarrow \prod_{i \in I} \coprod_{j \in J_i} \mathbb{C}(-, C_i)$$

where K is the set of all choice functions $f : I \longrightarrow \bigcup_{i \in I} J_i$, $f(i) \in J_i$. This set has cardinality less than α, under the assumption that α is strong limit. Each functor $\prod_{i \in I} \mathbb{C}(-, C_{f(i)})$ is α-generated (use conditions on $\mathbb{C}$). Thus,

$\prod_{i\in I} F_i$ is α-generated and consider the cover we constructed:

$$\coprod_{f\in K}\coprod_{c}\mathbb{C}(-,B_c)\longrightarrow\!\!\!\!\!\rightarrow\prod_{i\in I}F_i$$

where c ranges over an αm-limit for the diagram $i\mapsto C_{f(i)}$. We denote the coproduct on the left-hand side above by L. This L is a weak product of F_i's in α-fam($\mathbb{C}$): consider a cone

$$\coprod_{z\in Z}\mathbb{C}(-,X_z)\longrightarrow F_i$$

in α-fam($\mathbb{C}$), find a mediating

$$m:\coprod_{z\in Z}\mathbb{C}(-,X_z)\longrightarrow\prod_{i\in I}F_i$$

in $[\mathbb{C}^{op},\mathsf{Set}]$ and lift it along the cover of $\prod_{i\in I} F_i$ (we use here that representables are projective).

(b) Given a pair

$$\mathbb{C}(-,X)\overset{f}{\underset{g}{\rightrightarrows}}\coprod_{j\in J}\mathbb{C}(-,Y_j)$$

it is either equalized by the empty presheaf or f and g are represented by arrows $f_j,g_j:X\rightrightarrows Y_j$ into the same component of the coproduct, thus a weak equalizer for f and g in α-fam($\mathbb{C}$) is the αm-equalizer of the latter pair in $\mathbb{C}$. A weak equalizer of general pair

$$\coprod_{i\in I}\mathbb{C}(-,X_i)\overset{f}{\underset{g}{\rightrightarrows}}\coprod_{j\in J}\mathbb{C}(-,Y_j)$$

is given by the object $\coprod_{(i,k)}\mathbb{C}(-,Z_{(i,k(i))})$, where $\mathbb{C}(-,Z_k)$ is a weak equalizer for

$$\mathbb{C}(-,X_i)\overset{f_i}{\underset{g_i}{\rightrightarrows}}\coprod_{j\in J}\mathbb{C}(-,Y_j)$$

This is due to the fact that in an extensive category, such as in α-fam($\mathbb{C}$), a weak equalizer of a pair out of an α-small coproduct $f,g:\coprod_i X_i\rightrightarrows Y$ is the α-small coproduct of weak equalizers of $f\cdot\mathrm{in}_i, g\cdot\mathrm{in}_i: X_i\to X\rightrightarrows Y$.

(c) An αm-terminal object T of $\mathbb{C}$ is a weak terminal object when considered as an object of α-fam($\mathbb{C}$).

□

Corollary 4.2. *$\mathbb{C}$ is fm-complete if and only if* fam($\mathbb{C}$) *has weak finite limits.*

Corollary 4.3. *For α an inaccessible cardinal, $\mathbb{C}$ is αm-complete if and only if α-*fam($\mathbb{C}$) *has weak α-limits.*

Remark 4.4.

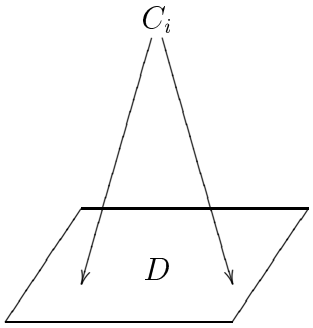
(1) Clearly, the following are equivalent:
(a) $\mathbb{C}$ is αm-complete.
(b) αm-limits of representables are α-generated in $[\mathbb{C}^{op}, \mathsf{Set}]$.

(a) $\Rightarrow$ (b): We know by Proposition 4.1 that α-colex($\mathbb{C}$) is closed in $[\mathbb{C}^{op}, \mathsf{Set}]$ under α-small limits of representables. Such limits are α-presentable functors, hence α-generated.

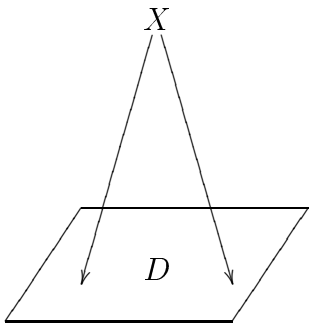
(b) $\Rightarrow$ (a): Let L be a limit of an α-small diagram $D : \mathbb{I} \longrightarrow \mathbb{C}$, considered as a diagram of representables in $[\mathbb{C}^{op}, \mathsf{Set}]$. Take an α-small cover

$$e : \coprod_{i \in I} \mathbb{C}(-, C_i) \longrightarrow\!\!\!\!\!\rightarrow L$$

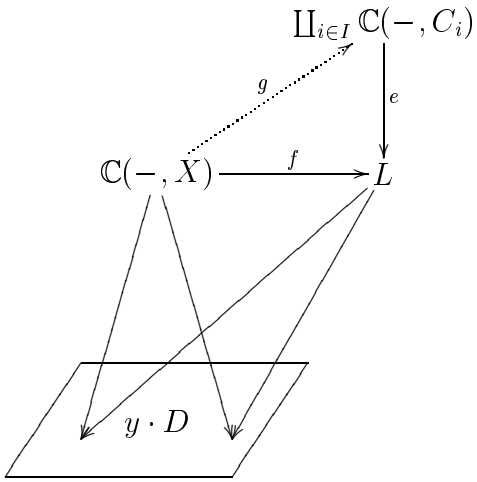
Thus we have an α-small set of cones



and this is an αm-limit of D in $\mathbb{C}$. Indeed, given a cone



consider it as a cone in α-colex($\mathbb{C}$)



Then the mediating f exists, since L is a limit. Now use projectivity of $\mathbb{C}(-,X)$ to derive the existence of g.

(2) If $\mathbb{C}$ is αm-complete, then α-products of representables are α-lucid in α-colex($\mathbb{C}$).

Use (1) to conclude that α-limits of representables are α-presentable. Thus, by (2), α-products of representables are α-lucid.

Lemma 4.5. *Let α be an uncountable cardinal and $\mathbb{C}$ be an α-complete category. Then a functor $P : \mathbb{C}^{op} \longrightarrow \mathsf{Set}$ is α-lucid if and only if it is α-presentable.*

PROOF. Take an α-presentable P and express it as an α-small colimit of representables

$$P = \operatorname{colim} \mathbb{C}(-, B_i)$$

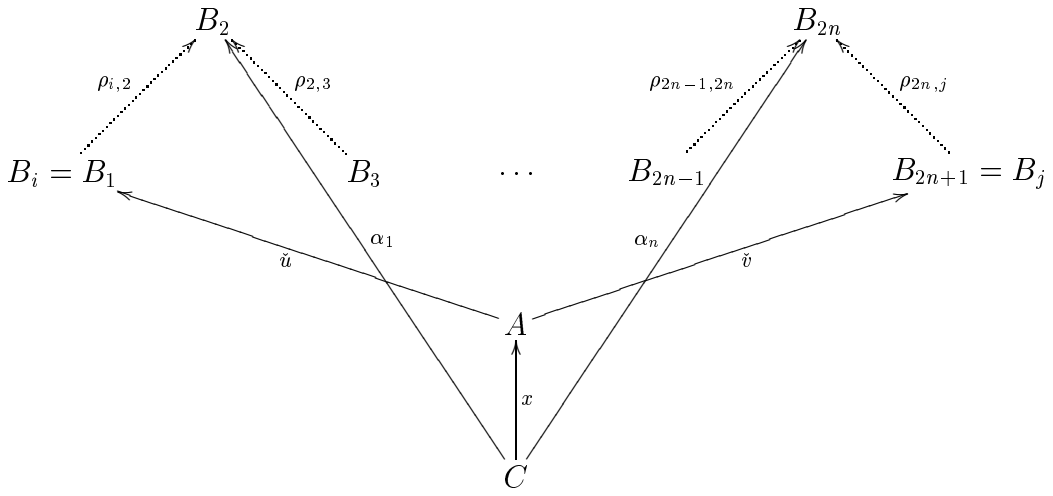
Now choose a pair $u, v : \mathbb{C}(-,A) \longrightarrow \operatorname{colim}\mathbb{C}(-,B_i)$ of morphisms and form an equalizer

$$E \longrightarrow \mathbb{C}(-,A) \underset{v}{\overset{u}{\rightrightarrows}} \operatorname{colim}\mathbb{C}(-,B_i)$$

The proof of [K$_1$], Theorem 3.3, shows that E is α-presentable.

In particular, the arrows u, v correspond, via Yoneda, to a pair of elements $\check{u}, \check{v} \in \operatorname{colim}\mathbb{C}(A, B_i)$. An arrow $x : C \longrightarrow A$ belongs to E at stage C if the composites $\check{u}\cdot x$, $\check{v}\cdot x$ represent equal elements in $\operatorname{colim}\mathbb{C}(C, B_i)$. Thus when B_i, B_j are not part of the same connected component of the diagram, the equalizer of u, v is the initial presheaf, which is α-presentable. So we assume that $\check{u}$, $\check{v}$ are connected by a zig-zag from B_i to B_j in the diagram and then

consider the diagram



For every finite diagram $\overline{B_z}$ consisting of some zig-zag connecting B_i and B_j, as well as $\check{u}$ and $\check{v}$, let $C_{z,n}$ be the objects of the α-small subcategory of $\mathbb{C}$ that forms the α-plurilimit for it, as provided by Proposition 3.4. In the proof of [K$_1$], Theorem 3.3, we exhibit a pair of natural maps

$$E \underset{e}{\overset{p}{\rightleftarrows}} \operatorname{colim}_z \operatorname{colim}_n \mathbb{C}(-, C_{z,n})$$

yielding E a retraction of the double colimit in the right hand side. The first colimit on the right hand side is taken over all possible zig-zags connecting B_i and B_j. Such zig-zags arise in the construction of the colimit $\operatorname{colim}\mathbb{C}(-, B_i)$ via coproducts and coequalizers, by forming the transitive hull of the reflective symmetric relation determined by a pair of parallel maps. Thus it is a countable colimit. As morphisms between zig-zags in the indexing category for that colimit are taken morphisms between their respective vertices satisfying the obvious commutativities. Also notice that every morphism between zig-zags induces morphisms of cones $C_{z,n} \to C_{z',n'}$ for any of the cones of the diagrams $\overline{D_z}$ and $\overline{D_{z'}}$. Thus E is α-presentable. □

Remark 4.6. In [K$_1$] it is erroneously claimed that the colimit over all zig-zags described above is finite. It is obvious that this can not be the case unless we have that transitive closures of reflexive symmetric relations between finite coproducts of representables are constructed in finitely many steps. This situation is exactly analyzed in the next section.

Proposition 4.7. *For any regular cardinal α, if α-colex($\mathbb{C}$) has α-small limits, then $\mathbb{C}$ has αm-limits.*

PROOF. This is the one direction of Theorem 3.3 in [K$_1$], where the author shows that, under the assumption that α-colex($\mathbb{C}$) is α-complete, $\mathbb{C}$ is

α-pluricomplete (stated as "coherent implies 2/3-SFP" there). Thus by Theorem 3.5, $\mathbb{C}$ is αm-complete. Of course the author there has not realized that the condition of being α-pluricomplete is of equal strength as being αm-complete. □

Theorem 4.8. *Let α be an uncountable regular cardinal. For an αm-complete category $\mathbb{C}$, α-colex($\mathbb{C}$) has finite limits. If moreover α is inaccessible, then α-colex($\mathbb{C}$) has α-small limits.*

PROOF. To prove the first claim notice the following: α-colex($\mathbb{C}$) has a terminal object, because the terminal object in $[\mathbb{C}^{op}, \mathsf{Set}]$ is α-presentable since it is isomorphic to the α-colimit $\mathrm{colim}_{\mathbb{I}}\, \mathbb{C}(-, P_i)$, where $P : \mathbb{I} \longrightarrow \mathbb{C}$ is an α-plurilimit of the empty diagram in $\mathbb{C}$. Further, α-colex($\mathbb{C}$) has binary products, because the product of two α-presentable functors in $[\mathbb{C}^{op}, \mathsf{Set}]$ is

$$\operatorname*{colim}_i \mathbb{C}(-, C_i) \times \operatorname*{colim}_j \mathbb{C}(-, D_j) \cong \operatorname*{colim}_i \operatorname*{colim}_j \Big(\mathbb{C}(-, C_i) \times \mathbb{C}(-, D_j)\Big)$$

and $\mathbb{C}(-, C_i) \times \mathbb{C}(-, D_j) \cong \mathrm{colim}_k\, \mathbb{C}(-, P_k)$ where $P : \mathbb{K} \longrightarrow \mathbb{C}$ is an α-plurilimit of the discrete diagram with vertices C_i, D_j in $\mathbb{C}$. Under the hypothesis of αm-completeness for $\mathbb{C}$, equalizers of pairs of parallel arrows from a representable functor to an α-presentable one are α-presentable, by the argument in Lemma 4.5. Finally, equalizers of parallel arrows between α-presentable objects are themselves α-presentable, as explained at the end of the proof of Theorem 3.3 in [K$_1$].

The proof of the second claim can be extracted from [F], Theorem 1.12 (equivalence of conditions (4) and (5) there). □

Of course, the above theorem applies to every α-complete category. Notice, however, that in the above proof we do not use fully the hypothesis that αm-limits exist for all α-small diagrams but only for finite ones. Therefore we have the following:

Corollary 4.9. *Let $\mathbb{C}$ have finite limits. Then, for an uncountable regular cardinal α, α-colex($\mathbb{C}$) has finite limits.*

Let us recall from [MR] the notion of an *α-pretopos*: It is an exact category with α-small sums which are disjoint and universal. Recall further from [K$_2$] that flatness of a functor on a small category is a property that, being expressible in geometric logic, it can be interpreted in any category equipped with a Grothendieck topology. In particular an α-pretopos comes equipped with the α-precanonical topology, i.e., the topology whose basic covering families are the epimorphic families of size $< \alpha$.

Proposition 4.10. *Let $\mathbb{C}$ be a left exact category, $\mathbb{E}$ be an α-pretopos, where α is an uncountable regular cardinal, and $F : \mathbb{C} \longrightarrow \mathbb{E}$ a functor that is flat with respect to the α-precanonical topology of the pretopos $\mathbb{E}$. Then the left Kan extension $\mathrm{Lan}_\eta F : \alpha\text{-colex}(\mathbb{C}) \longrightarrow \mathbb{E}$ of F along the inclusion $\eta : \mathbb{C} \longrightarrow \alpha\text{-colex}(\mathbb{C})$ is a left exact functor.*

PROOF. Notice first of all the left Kan extension exists and is computed by the usual colimit formula, because the indexing category of the colimit, namely η/X, X in α-colex($\mathbb{C}$), contains a final α-small subcategory. Next Lemma 3.3 of [K_2] applies in this case. The lemma in question requires that a certain technical condition is fulfilled, namely that the colimits entering in the calculation of the left Kan extension are *postulated*. This follows essentially from Lemma 2.1 in [AK], by an appropriate modification from the ∞ case covered there to the case of a cardinal α as in the hypothesis here. The modification is possible because the transitive closure of a reflexive, symmetric relation, needed to compute a coequalizer in an α-pretopos, is a countable union and as such lives in the α-pretopos. □

Corollary 4.11. *Let $F : \mathbb{C} \longrightarrow \mathbb{D}$ be a left exact functor between left exact categories. Then, for an uncountable regular cardinal α, the induced $F^* : \alpha\text{-colex}(\mathbb{C}) \longrightarrow \alpha\text{-colex}(\mathbb{D})$ preserves finite limits (as well as α-colimits).*

PROOF. The induced F^* is the left Kan extension of $\eta_{\mathbb{D}} \cdot F$ along the inclusion $\eta_{\mathbb{C}} : \mathbb{C} \longrightarrow \alpha\text{-colex}(\mathbb{C})$. It is immediate to see that $\eta_{\mathbb{D}} \cdot F$, being a composite of left exact functors, is flat for the α-precanonical topology. □

The above corollary means that the α-cocompletion (pseudo)monad

$$\alpha\text{-colex} : \mathsf{Cat} \longrightarrow \mathsf{Cat}$$

on the category of small categories lifts to a (pseudo)monad

$$\alpha\text{-colex} : \mathsf{Lex} \longrightarrow \mathsf{Lex}$$

on the category of small categories with finite limits. By results on the theory of monads this amounts to the existence of a distributive law

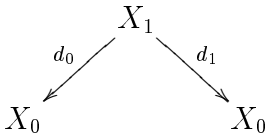
$$\mathrm{lex} \cdot \alpha\text{-colex} \Longrightarrow \alpha\text{-colex} \cdot \mathrm{lex} : \mathsf{Cat} \longrightarrow \mathsf{Cat}$$

5 Finite Completeness of colex($\mathbb{C}$)

In this section we investigate finite completeness of colex($\mathbb{C}$), the free completion of $\mathbb{C}$ under finite colimits. To be able to state our main result we need to consider transitive hulls of reflexive symmetric graphs in categories with finite weak limits.

Definition 5.1. Let $\mathbb{X}$ be a category.

(1) A *graph* X in $\mathbb{X}$ is a span



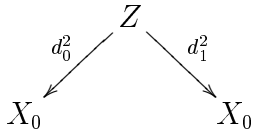
in $\mathbb{X}$. We call X_0 an *object of vertices* of X, X_1 an *object of edges* of X and d_0, d_1 are *domain* and *codomain* maps of X, respectively.

A graph X is *reflexive*, if there is a *splitting map* $s : X_0 \longrightarrow X_1$ satisfying equations $d_0 \cdot s = 1_{X_0}$ and $d_1 \cdot s = 1_{X_0}$.

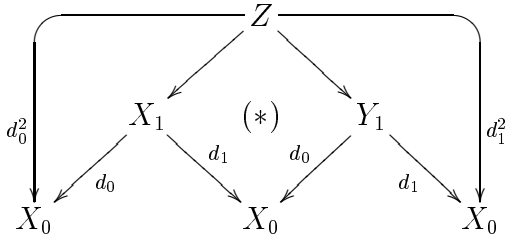
A graph X is *symmetric*, if there is a *twist map* $t : X_1 \longrightarrow X_1$ satisfying equations $t \cdot d_0 = d_1$ and $t \cdot d_1 = d_0$.

(2) Given graphs X and Y, having the same object of vertices X_0, then a *morphism* from X to Y is $f : X_1 \longrightarrow Y_1$ respecting domains and codomains, i.e., such that equations $d_0 \cdot f = d_0$ and $d_1 \cdot f = d_1$ hold.

(3) Suppose $\mathbb{X}$ has weak pullbacks. Given graphs X and Y, having the same object of vertices X_0, then their *composite* $X;Y$ is a graph



obtained as follows:



where the square $(*)$ is a weak pullback.

For a graph X, we use the notation $X^{(1)} = X$ and $X^{(n+1)} = X^{(n)};X$.

(4) A category $\mathbb{X}$ with weak pullbacks is said to satisfy condition *BTH* (*bounded transitive hulls* of reflexive symmetric graphs), provided that

for each reflexive symmetric graph X there exists n such that for all $m > n$ there exists a morphism of graphs $f_m : X^{(m)} \longrightarrow X^{(n)}$, i.e., such that the equations $d_0^n \cdot f_m = d_0^m$ and $d_1^n \cdot f_m = d_1^m$ hold.

Theorem 5.2. *For a category $\mathbb{C}$, the following are equivalent:*

(1) $\mathrm{colex}(\mathbb{C})$ *has finite limits.*
(2) $\mathbb{C}$ *is fm-complete and* $\mathrm{fam}(\mathbb{C})$ *satisfies BTH.*
(3) $\mathrm{colex}(\mathbb{C})$ *is equivalent to the category* $[\mathbb{C}^{op}, \mathsf{Set}]_{coh}$ *of coherent objects in* $[\mathbb{C}^{op}, \mathsf{Set}]$.
(4) The category $[\mathbb{C}^{op}, \mathsf{Set}]_{coh}$ *is closed under coequalizers in* $[\mathbb{C}^{op}, \mathsf{Set}]$.

PROOF. (3) $\Leftrightarrow$ (4) by [SGA4], Exposé IV, Corollaire 1.25.

(3) $\Rightarrow$ (1): Since coherent objects are closed under finite limits in $[\mathbb{C}^{op}, \mathsf{Set}]$.

(1) $\Rightarrow$ (3): Every F in $\mathrm{colex}(\mathbb{C})$ is finitely presentable in $[\mathbb{C}^{op}, \mathsf{Set}]$. Then, for each pair S, T of finitely presentable objects, the pullback

$$\begin{array}{ccc} P & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle v} \\ S & \xrightarrow[u]{} & F \end{array}$$

is a finitely presentable object for each pair u, v of morphisms. Then P is a coherent object by 9.5.1 of [MR] and therefore $\mathrm{colex}(\mathbb{C}) \simeq [\mathbb{C}^{op}, \mathsf{Set}]_{coh}$.

(2) $\Rightarrow$ (1): We know by Proposition 4.1 that $\mathbb{X} = \mathrm{fam}(\mathbb{C})$ has weak finite limits and it has finite coproduct by definition. We use the result of [PR] that, for such categories $\mathbb{X}$, condition BTH is equivalent to $\mathrm{rsc}(\mathbb{X}) \simeq \mathrm{ex}(\mathbb{X})$ (closure of $\mathbb{X}$ under coequalizers of reflexive symmetric pairs stands on the left and exact completion stands on the right). Thus,

$$\mathrm{rsc}(\mathrm{fam}(\mathbb{C})) \simeq \mathrm{ex}(\mathrm{fam}(\mathbb{C}))$$

By a general result of Pitts (see [BC]), $\mathrm{rsc}(\mathrm{fam}(\mathbb{C})) \simeq \mathrm{colex}(\mathbb{C})$ holds. Putting things together, we obtain $\mathrm{colex}(\mathbb{C}) \simeq \mathrm{ex}(\mathrm{fam}(\mathbb{C}))$, and the category on the right-hand side has finite limits, being an exact completion.

(1) $\Rightarrow$ (2): Since $\mathrm{colex}(\mathbb{C})$ has finite limits, it follows by Proposition 4.1 that $\mathbb{C}$ is fm-complete. By [SGA4], Exposé VI, Exercise 2.17(c), the category $[\mathbb{C}^{op}, \mathsf{Set}]$ is a coherent topos, hence $[\mathbb{C}^{op}, \mathsf{Set}]_{coh}$ is a pretopos, therefore an exact category. By (1) $\Leftrightarrow$ (3) we conclude that $\mathrm{colex}(\mathbb{C}) \simeq [\mathbb{C}^{op}, \mathsf{Set}]_{coh}$ is an exact category.

Now use the fact that fam($\mathbb{C}$) is a projective cover of colex($\mathbb{C}$), thus, by [CV] we know that fam($\mathbb{C}$) has weak limits and ex(fam($\mathbb{C}$)) $\simeq$ colex($\mathbb{C}$). Use [PR] to conclude that BTH holds in fam($\mathbb{C}$). $\square$

Remark 5.3. In the theory of additive locally presentable categories the term "locally coherent" is used in order to describe the situation where finitely presentable objects are closed under finite limits. In the additive setting this amounts to the condition that finitely generated subobjects of finitely presentable ones are themselves finitely presentable. Notice that, generally, in an exact locally finitely presentable category the closure of finitely presentable objects under finite limits implies the latter condition. For, if $F \hookrightarrow P$ is an finitely generated subobject of a finitely presentable one, then we can produce a finite presentation of F as follows: Let $Q \twoheadrightarrow F$ be a finitely presentable cover of F. Consider its kernel pair $K \rightrightarrows Q \twoheadrightarrow F$. It is the same as the kernel pair of $Q \twoheadrightarrow F \hookrightarrow P$. By the assumption that finitely presentable objects are closed under finite limits, K is finitely presentable. By exactness, $K \rightrightarrows Q \twoheadrightarrow F$ is a coequalizer. Thus F is finitely presentable. In particular, if a category $\mathbb{C}$ satisfies the equivalent conditions above then the presheaf category $[\mathbb{C}^{op}, \mathsf{Set}]$ is "locally coherent" in the sense of additive category theory.

Example 5.4. An example of a category $\mathbb{C}$ with finite limits such that colex($\mathbb{C}$) does not have finite limits.

Let X be an infinite set and let R be a binary reflexive and symmetric relation on X such that the sequence

$$R \subseteq R \circ R \subseteq R \circ R \circ R \subseteq \dots$$

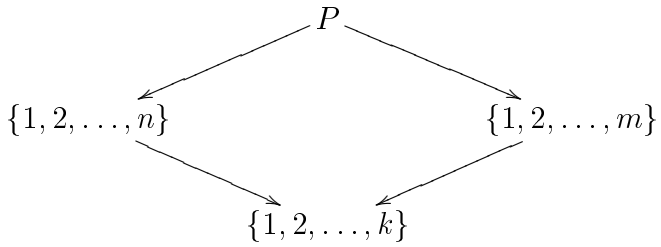
does not stabilize. Let $\mathbb{C}$ be a full subcategory of Set containing X and all R^n, $n \geq 1$, and such that $\mathbb{C}$ is closed under finite limits in Set. Then fam($\mathbb{C}$) does not satisfy BTH, hence colex($\mathbb{C}$) does not have finite limits.

Remark 5.5. We formulate now the BTH condition in fam($\mathbb{C}$) elementarily in $\mathbb{C}$. To be able to do that we show first how weak pullbacks are computed in fam($\mathbb{C}$), provided $\mathbb{C}$ is fm-complete.

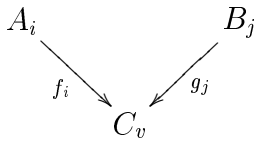
Given a cospan

$$\begin{array}{ccc} \langle A_1, A_2, \dots, A_n \rangle & & \langle B_1, B_2, \dots, B_m \rangle \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle g} & \\ & \langle C_1, C_2, \dots, C_k \rangle & \end{array} \tag{5.1}$$

in fam($\mathbb{C}$), we form first a pullback of indexing sets in Set:



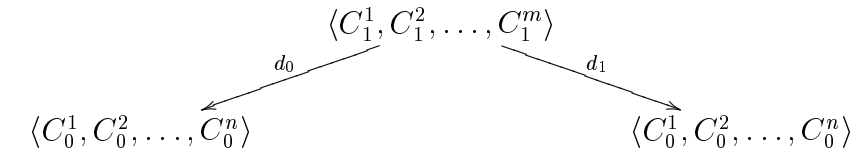
i.e., P consists of all pairs (i, j) such that there is a cospan



(5.2)

in $\mathbb{C}$. This finite set P is the indexing set of a weak pullback of (5.1) above. We denote this weak pullback by $\langle D_{(i,j)} \mid (i, j) \in P \rangle$ where $D_{(i,j)}$ is an fm-limit of (5.2) in $\mathbb{C}$. (The pullback projections are defined in an obvious way.)

A graph (and its reflexivity, symmetry and BTH)



in fam($\mathbb{C}$) is translated into $\mathbb{C}$ as follows:

(1) Think of $\langle C_0^1, C_0^2, \ldots, C_0^n \rangle$ as of a "partition" of an "object C_0 of vertices" in $\mathbb{C}$ and of $\langle C_1^1, C_1^2, \ldots, C_1^m \rangle$ as of a "partition" of an "object C_1 of edges" in $\mathbb{C}$, where C_1^j contains edges of colour j. A graph then assigns a domain and a codomain vertex to each edge, partition of C_0 plays no rôle so far.

(2) Reflexivity $s : C_0 \longrightarrow C_1$ produces a loop on each vertex, and all loops in C_0^i have colour $s(i)$.

(3) Symmetry $t : C_1 \longrightarrow C_1$ tells us that for each edge $v \longrightarrow w$ of colour i there is an edge $w \longrightarrow v$ of colour $t(i)$.

(4) The "second power" of a graph creates colour (i, j) for "composable" pairs of edges of colours i and j. All loops in C_0^i remain to be of colour i (of colour (i, i), more precisely).

"Higher powers" of a graph have a similar description.

(5) The existence of n, such that for all $m > n$ there exists a graph homomorphism $f_m : C_m \longrightarrow C_n$ says that if a path of colour $(i_1, i_2, \ldots, i_m)$ appears in C_m, there must have been a path of colour $(j_1, j_2, \ldots, j_n)$ in C_n connecting the same pair of vertices.

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