Sifted Inductive Completion over Cartesian Closed Bases

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Sifted colimits.

A category is sifted if colimits indexed by it commute in Set with binary products. Equivalently, if

1. For any two objects there exists a cospan to a third one
2. Any two cospans from two given objects are connected by a zig-zag.

Sifted flatness

A functor $F\colon \mathcal{A} \longrightarrow \mathrm{Set}$ is sifted flat if its left Kan extension along the Yoneda embedding preserves finite products.

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Theorem: A functor $F\colon \mathcal{A} \longrightarrow \mathrm{Set}$ is sifted flat iff the dual of its category of elements is sifted.

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Sifted inductive completion

$\mathrm{Sind}(\mathcal{A})$ is the free cocompletion of $\mathcal{A}$ under sifted colimits. It can be described as the closure of representables in $[\mathcal{A}^{op}, \mathrm{Set}]$ under sifted colimits.

Varying the base category

For certain purposes (e.g homotopical algebra) we may want category to mean category enriched over a symmetric monoidal closed $\mathscr{V}$ (e.g $\mathscr{V}$=simplicial sets). In particular:

A convenient setting to study cocompletions of enriched categories under sifted colimits is that of a cartesian closed base $\mathcal{V}$ which is strongly lfp as closed category. This means that

1. Has a set of dense generators G that are strongly f.p, i.e $\mathrm{colim}_d \mathcal{V}(G, V_d) \cong \mathcal{V}(G, \mathrm{colim}_d V_d)$ is an iso whenever V_d is a sifted diagram.
2. I is sfp
3. $V_1 \times V_2$ is sfp, whenever V_1, V_2 are sfp.

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1. Presheaf categories $[\mathcal{G}^{op}, \mathrm{Set}]$, when $\mathcal{G}$ has finite products
2. or, more generally, has the property that the terminal presheaf and the product of two representables is a finite coproduct of representables
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Sifted weights

A weight $w\colon \mathcal{D}^{op} \longrightarrow \mathcal{V}$ is sifted if $Lan_Y w$ preserves

1. finite (conical) products and
2. cotensors with sfp objects of $\mathcal{V}$.

Sifted inductive completion

For a $\mathcal{V}$-category $\mathcal{A}$ we define $\mathrm{Sind}(\mathcal{A})$ to be the closure of representables in $[\mathcal{A}^{op}, \mathcal{V}]$ under sifted weighted colimits.
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Assume that $\mathcal{V}$ is strongly lfp as a (cartesian) closed category, $V\colon \mathcal{V}_o \longrightarrow \mathrm{Set}$ preserves colimits (as it is the case when $\mathcal{V}_o$ is presheaves on a category with terminal object). In this setting:

Lemma 1: If $F\colon \mathcal{A} \longrightarrow \mathcal{V}$ is sifted flat, $(\mathrm{elts} VF_o)^{op}$ is a sifted category.

Lemma 2: If $\mathcal{A}$ admits cotensors with sfp objects and $F\colon \mathcal{A} \longrightarrow \mathcal{V}$ is sifted flat, then F is a conical sifted colimit of representables. (When $\mathcal{V}$ is a presheaf category cotensors with representables suffice.)

Lemma 3: Conical sifted colimits of representables are sifted flat.

Corollary: If $\mathscr{A}$ admits tensors with sfp objects then $(\mathrm{Sind}\mathcal{A})_o \cong \mathrm{sind}(\mathcal{A}_o)$.

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Proof of Lemma 1:

Consider the product of two representables $\mathcal{A}_o(-, A_1) \times \mathcal{A}_o(-, A_2) \colon \mathcal{A}_o^{op} \longrightarrow \mathrm{Set}$. We want it to be preserved by $\mathrm{Lan}_y VF_o$

1. $\mathrm{Lan}_y VF_o(\mathcal{A}_o(-, A_1) \times \mathcal{A}_o(-, A_2)) \cong$
2. $\int^{A \in \mathcal{A}_o} \mathcal{A}_o(A, A_1) \times \mathcal{A}_o(A, A_2) \times VF_o(A) \cong$
3. $\int^{A \in \mathcal{A}_o} V\mathcal{A}(-, A_1)_o(A) \times V\mathcal{A}(-, A_2)_o(A) \times VF_o(A) \cong$
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Almost identical to [Kelly, Structures defined by finite limits in the enriched context], Prop. 6.9. Hinges on the fact that F preserves cotensors with sfp objects of $\mathcal{V}$. With that we show that F is a conical colimit of representables, indexed by $(\mathrm{elts}\, VF_o)^{op}$.

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Theorem: Let $\mathscr{A}$ be a $\mathscr{V}$-category that has tensors by sfp objects in $\mathscr{V}$ (representables would suffice if $\mathcal{V}$ where presheaves). Then $\mathscr{A}$ is Sind-Lim-multicomplete iff $\mathrm{Sind}\mathscr{A}$ is complete.

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Suffices to show:

1. The underlying ordinary category $\mathrm{Sind}(\mathscr{A})_o$ has small limits.
2. $\mathrm{Sind}(\mathscr{A})$ has tensors by representables (necessary in order to show that small limits in $\mathrm{Sind}(\mathscr{A})_o$ are actual conical limits in $\mathrm{Sind}(\mathscr{A})$).
3. $\mathrm{Sind}(\mathscr{A})$ has cotensors.

Small limits exist in the underlying category

1. because $\mathrm{Sind}(\mathscr{A})$ has limits of representables (as J.V's talk), hence $\mathrm{Sind}(\mathscr{A})_o$ has limits of representables, hence $\mathcal{A}_o$ is sind-Lim-multicomplete as an ordinary category, hence $\mathrm{sind}(\mathscr{A}_o)$ has all the limits that $\mathrm{sind}(\mathrm{Lim}\mathscr{A}_o)$ has (again J.V's talk), but the latter is complete by [Adamek, Lawvere, Rosický, How algebraic is algebra?].

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2. Define the tensor of a representable $\mathcal{A}(-, A)$ by an sfp K as $K \otimes \mathcal{A}(-, A) = \mathcal{A}(-, K \otimes A)$. Then extend by (conical) sifted colimits.
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Corollary: For $\mathcal{V}$, V as above and a $\mathcal{V}$-category $\mathcal{K}$ the following are equivalent:

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Notice that, for a Cauchy complete $\mathcal{G}$, an object in $[\mathcal{G}^{op}, \mathrm{Set}]$ is sfp iff it is a finite coproduct of representables. Thus simplicial sets are not strongly lfp as a closed category, or else the square would have to be the coproduct of two triangles.
Symmetric simplicial sets [Rosický, Tholen, Left determined model categories] may be a better option for studying homotopy varieties.

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