Left Kan Extensions Preserving Finite Products

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1 Introduction

In many situations in mathematics we want to know whether the left Kan extension $\mathrm{Lan}_j F$ of a functor $F : \mathcal{C} \to \mathcal{E}$, along a functor $j : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ is a small category and $\mathcal{E}$ is locally small and cocomplete, preserves the limits of various types Φ of diagrams. The answer to this general question is reducible to whether the left Kan extension $\mathrm{Lan}_y F$ of F, along the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ preserves those limits (see [12] §3). Such questions arise in the classical context of comparing the homotopy of simplicial sets to that of spaces but are also vital in comparing, more generally, homotopical notions on various combinatorial models (e.g simplicial, bisimplicial, cubical, globular, etc, sets) on the one hand, and various "realizations" of them as spaces, categories, higher categories, simplicial categories, relative categories etc, on the other (see [8], [4], [15], [2]).

In the case where $\mathcal{E} = \mathbf{Set}$ the answer is classical and well-known for the question of preservation of all finite limits (see [5], Chapter 6), the question of finite products (see [1]) as well as for the case of various types of finite connected limits (see [11]). The answer to such questions is also well-known in the case $\mathcal{E}$ is a Grothendieck topos, for the class of all finite limits or all finite connected limits (see [14], chapter 7 § 8 and [9]).

We take up here the question of preservation of finite limits by Kan extensions into more general cocomplete categories $\mathcal{E}$. In particular in this work we focus on the case of finite products. We leave the case of connected limits for a subsequent work. In the context of sets or, more generally, Grothendieck toposes the well-known answer to the question has to do with the filteredness of the category of elements of the functor F. When we deal with finite products alone, in sets, the answer is also given in similar terms; the weaker condition of siftedness of the category of elements is necessary and sufficient. These notions though are hard to state for functors with values into a topos: The internal logic of the topos has to be invoked. Given this difficulty some authors prefer to talk about variations of those conditions that involve the composites with forgetful functors (when available) or with representable functors (otherwise) into sets. These are essentially interpretations of the standard filteredness notions with respect to a certain strong site structure on the cocomplete category. One point that we try to make here is that other weaker structures may be relevant to the limit preservation question. In particular, the crude notions mentioned above may fail to capture cases where the preservation of finite limits by the left Kan extension is well-known, e.g by geometric realization of simplicial sets in Kelley speces. The topos-theoretic context, because of the exactness conditions that it supplies, conceals another condition that is relevant to our problem: The colimits used in the calculation of the left Kan extension of a product have to behave well. This is a fundamental observation due to Anders Kock ([12]) and requires ideas of categorical logic and a site structure on the cocomplete category for its statement. A. Kock uses Diaconescu's theorem for toposes and with the aid of a metamathematical argument he derives preservation of finite limits by flat functors into suitable subcanonical sites. Our approach differs in that we give a completely elementary proof, not depending on Diaconescu's theorem. Instead, Diaconescu's theorem will be derived as a corollary to our work (including the subsequent one on connected finite limits). Moreover, strictly speaking, no "Diaconescu's theorem" had been available for finite-product-preserving left Kan extensions into toposes. This is our Corollary 4.5. In more general situations (functors with values in subcanonical sites) we also provide a partial answer to whether functors with finite-product-preserving left Kan extensions are precisely the sifted-flat ones.

Organization of the paper In Section 2. we recall the notion of siftedness, point out that it is expressible in geometric logic and hence interpretable in locally small categories with a site structure and give (non-) examples of functors that fail to "have a sifted category of elements" (in the crude sense), yet their left Kan extension preserves finite products.

In Section 3. we recall the notion of postulated colimit and explain why certain colimits used for the calculation of geometric and categorical realizations of certain simplicial, bisimplicial and globular sets are postulated with respect to certain site structures.

In Section 4. we prove our main results: Functors that have finite-product-preserving left Kan extensions are precisely those that have a suitably sifted category of elements, provided that certain colimits are postulated in the recipient category. In particular a functor into a Grothendieck topos has a finite-product-preserving left Kan extension if and only if its category of elements is sifted in the internal logic of the topos.

As said, the conditions involved in answering the problems studied here are renderings of familiar notions, when interpreted via sheaf semantics (interpretation of geometric sentences relative to a notion of covering, as it is exposed in standard references, e.g [14], [13]). Their statement can be given in completely elementary terms, though it is not the most economical one. Hence no familiarity with sheaf semantics is essentially required.

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2 Sifted categories and sifted-flat functors

2.1 Definition. A category $\mathcal{C}$ is called sifted if for every pair A, B of objects of $\mathcal{C}$ the category $(A, B) \downarrow \mathcal{C}$ of all (A, B)-cospans is connected. More precisely the following two conditions are satisfied:

1. $(A, B) \downarrow \mathcal{C}$ is nonempty, i.e, there exists at least one object $X \in \mathcal{C}$ and morphisms $A \to X$ and $B \to X$.

2. Every two objects in $(A, B) \downarrow \mathcal{C}$ are connected via a zig-zag, i.e, for each two objects in the category $(A, B) \downarrow \mathcal{C}$, there exists a zig-zag in this category connecting them.

2.2 Remark. Usually a sifted category is defined as a category $\mathcal{C}$ with the property that, colimits over $\mathcal{C}$ commute in **Set** with finite products (see [1]). The equivalence of the two notions is of course non-trivial. Towards the end of the last section we explain how this equivalence can be derived from our analysis.

2.3 Remark. A (small) category $\mathcal{C}$ with finite coproducts is sifted

2.4 Definition. A functor $F : \mathcal{C} \to$ **Set** is called **sifted flat** if the dual of the category of elements of F is sifted. More explicitly if

1. For every $C_1 \in \mathcal{C}$ and $x_1 \in FC_1$, $C_2 \in \mathcal{C}$ and $x_2 \in FC_2$, there exists $C_3 \in \mathcal{C}$, $x_3 \in FC_3$ and arrows $f : C_3 \to C_1$, $g : C_3 \to C_2$, such that $Ff(x_3) = x_1$ and $Fg(x_3) = x_2$.

$$\begin{tikzcd} & (C_3, x_3) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ (C_1, x_1) & & (C_2, x_2) \end{tikzcd}$$

   and

2. For each A, B in $\mathcal{C}$ and for every X in $\mathcal{C}$, $x \in FX$, and X' in $\mathcal{C}$, $x' \in FX'$ and diagram in $\mathcal{C}$

$$\begin{tikzcd}
 & (X,x) \arrow[dl, "f"'] \arrow[dr, "g"] & \\
(A, Ff(x)) & & (B, Fg(x)) \\
 & (X',x') \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

such that $Ff(x) = Ff'(x')$ and $Fg(x) = Fg'(x')$, there exists a zig-zag connecting X, X', as in the diagram below, such that all the internal triangles in it are commutative and, moreover, there exists elements $x_1 \in FX_1$, $x_2 \in FX_2, ..., x_n \in FX_n$ such that

$$Fd_1^0(x_1) = x',\ Fd_1^1(x_1) = x_2,\ ...,\ Fd_n^0(x_n) = x_{n-1},\ Fd_n^1(x_n) = x.$$

$$\begin{tikzcd}
 & & X \arrow[ddll, "f"'] \arrow[ddrr, "g"] & & \\
 & & X_n \arrow[u, "d_n^1"] \arrow[dll, "f_n"'] \arrow[drr, "g_n"] \arrow[d, dotted, no head] & & \\
A & & X_3 \arrow[ll, "f_3"] \arrow[rr, "g_3"'] \arrow[d, "d_3^0"'] & & B \\
 & & X_2 \arrow[ull, "f_2"] \arrow[urr, "g_2"'] & & \\
 & & X_1 \arrow[u, "d_1^1"] \arrow[uull, "f_1"] \arrow[uurr, "g_1"'] \arrow[d, "d_1^0"'] & & \\
 & & X' \arrow[uuull, "f'"] \arrow[uuurr, "g'"'] & &
\end{tikzcd}$$

One then has ([1], Theorem 2.6)

2.5 Theorem. *A functor $F : \mathcal{C} \to$* **Set** *is sifted flat if and only if its left Kan extension* $\mathrm{Lan}_y F$*, along the Yoneda embedding of $\mathcal{C}$ preserves finite products.*

2.6 Remark. When one deals with a functor $F:\mathcal{C}\to\mathcal{E}$, into a locally small, cocomplete category with finite products, then a similar characterization is missing. Some authors deal with flatness notions in this more general context in a rather inadequate way: F is (or ought to be) sifted-flat when for each object $E\in\mathcal{E}$, the functor $\mathcal{E}(E,F-):\mathcal{C}\to\mathbf{Set}$ is sifted-flat. This is partly justified by the fact that a functor $F:\mathcal{C}\to\mathcal{E}$, where $\mathcal{C}$ and $\mathcal{E}$ have finite products, preserves finite products if and only if it is sifted-flat in this sense. But in the absense of finite products in $\mathcal{C}$ things do not work that well, as the example of the following paragraph shows.

Simplicial sets and their realization: Let $\mathbf{\Delta}$ be the category whose objects are finite, totally ordered sets

$$[n]=\{0,\,1,\ldots,\;n\}$$

and morphisms are order preserving functions. Consider the functor

$$U:\mathbf{\Delta}\to\mathbf{Cat}$$

from $\mathbf{\Delta}$ to the category of small categories that sees a linear order as a category.

Claim 1: U is not sifted-flat, in the sense that it is not true that, for any small category $\mathcal{C}$, the functor

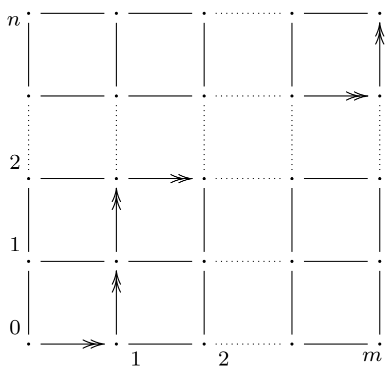
$$\mathbf{Cat}(\mathcal{C},U-)\colon\mathbf{\Delta}\to\mathbf{Set}$$

has a sifted category of elements.

This will take a while to explain. It depends on the following proposition, which is an explication of [8], 5.5, in view of [10], Proposition 3.4.

2.7 Proposition. *In the category $\mathbf{\Delta}$, for any two objects $[m]$, $[n]$, there is a finite family of cones over them, all of them having as vertex the ordinal $[m+n]$, with the property that any other cone factors through one in that family.*

Proof: Given $[m]$, $[n]$ in $\mathbf{\Delta}$ consider the family of pairs of surjective, order-preserving maps $\alpha\colon[m+n]\to[m]$, $\beta\colon[m+n]\to[n]$. They correspond to maximum length paths on the $[m]\times[n]$ grid on the plane, where motion is allowed only upwards and to the right, since α and β are order-preserving:



So the path depicted above corresponds to the maps α and β having $\alpha(0) = 0$, $\alpha(1) = \alpha(2) = 1$, ..., $\alpha(m+n-2) = m-1$, $\alpha(m+n-1) = \alpha(m+n) = m$ while $\beta(0) = \beta(1) = 0$, $\beta(2) = 1$, $\beta(3) = 2$, ..., $\beta(m+n-2) = \beta(m+n-1) = n-1$, $\beta(m+n) = n$. There are obviously $\binom{m+n}{n}$ such cones.

Now given any cone $\gamma\colon [k] \to [m]$, $\delta\colon [k] \to [n]$, there is a set of points $(\gamma(i), \delta(i))$, $i \in [k]$, inside the $[m] \times [n]$-grid. Extend that set of points into a maximal path in any possible way. This extension specifies a factorization through a cone with vertex $[m+n]$ so that the given one factors through that. The desired factorization is $\varepsilon\colon [k] \to [m+n]$ given by

$$\varepsilon(i) = \text{ the number of position of } (\gamma(i), \delta(i)) \text{ inside the maximal path} \quad \blacksquare$$

2.8 Proposition. *For any finite diagram in the category* $\mathbf{\Delta}$ *there is a finite family of cones over it with the property that any other cone factors through one in that family. Moreover, for any finite diagram* $D\colon \mathcal{D} \to \mathbf{\Delta}$*, the limit of the diagram of representables* $y \cdot D\colon \mathcal{D} \to \mathbf{\Delta} \to \mathbf{Sset}$ *in simplicial sets is isomorphic to a finite colimit of representables.*

Proof: For the first claim notice that $\mathbf{\Delta}$ has a terminal object and that a pair of parallel arrows either has an equalizer (if they agree somewhere) or otherwise they have an empty (hence finite) set of cones. Then we conclude from [3], Theorem 2.1. For the second claim, we conclude from [10], Theorem 3.5. $\blacksquare$

Returning to the proof of **Claim 1**, consider $[1] \times [1]$ with the pointwise order, viewed as a category, and take the following diagram in **Cat**

$$\begin{tikzcd} & {[1] \times [1]} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ {[1] = U([1])} & & {[1] = U([1])} \end{tikzcd}$$

If U was sifted-flat in the above sense, there would have to be an ordinal $[n]$, an order preserving function $[1] \times [1] \to [n]$ and two order preserving functions $[n] \rightrightarrows [1]$ such that in the following diagram the two triangles commute.

$$\begin{tikzcd}
 & {[1] \times [1]} \arrow[ddl, "p_1"'] \arrow[d, dotted, "h"] \arrow[ddr, "p_2"] & \\
 & {[n]} \arrow[dl, dotted, "f_1"] \arrow[dr, dotted, "f_2"'] & \\
{[1]} & & {[1]}
\end{tikzcd}$$

But, by Proposition 2.7 the desired factorization through an ordinal $[n]$ should eventually factorize through the ordinal $[2]$, so we are searching for a factorization as indicated in the following diagram

$$\begin{tikzcd}
 & {[1] \times [1]} \arrow[ddl, "p_1"'] \arrow[d, dotted, "h"] \arrow[ddr, "p_2"] & \\
 & {[2]} \arrow[dl, dotted, "f_1"] \arrow[dr, dotted, "f_2"'] & \\
{[1]} & & {[1]}
\end{tikzcd}$$

where f_1, f_2 are surjective and order-preserving. By direct inspection we see that there is no h such that the two triangles in the above diagram commute. This proves Claim 1. Yet,

Claim 2: The left Kan extension of U along the Yoneda embedding (a.k.a the categorical realization of simplicial sets) preserves finite products.

This can be easily seen, taking into account that the right adjoint to the categorical realization (a.k.a the nerve functor) is fully faithful, its image contains the representables and **Cat** is cartesian closed. Nevertheless, it can be derived as a corollary to the results in the next two sections.

Formulas for Flatness What resolves the "contradiction" in the above example? Topos theory indicates the answer: When dealing with flatness conditions one should take into account the local character of these conditions. In [14], chapter 7, we see that the key concept (e.g in proving Diaconescu's theorem) of filtering functor arises when one restates the usual flatness conditions "up to epi covering". This means that certain logical formulas are interpreted with respect to the canonical site structure of the topos. On the other hand, the version of sifted-flatness discussed in the previous paragraph amounts to the interpretation of the relevant conditions with respect to the coarsest site structure (where only isomorphisms cover).

More precisely, let $\mathcal{C}$ be a small category and consider the language $\mathfrak{L}_{\mathcal{C}}$ for "functors on $\mathcal{C}$": it has one sort C for each object of $\mathcal{C}$, and one function symbol $f\colon C \to C'$, for each arrow in $\mathcal{C}$, respectively. Structures of this language in sets are sorted sets $F(C)$ (one for each C), equipped with functions $Ff\colon F(C) \to F(C')$. Axioms in this language account for the functoriality of this assignment, hence models in sets correspond bijectively to functors $F\colon \mathcal{C} \to \mathbf{Set}$. The same applies to models of the same theory into any category.

Furthermore the two conditions describing the notion of a sifted-flat functor have a description in $\mathfrak{L}_{\mathcal{C}}$ sentences that are geometric, hence they can be interpreted in a sound way in any category equipped with a Grothendieck topology. In particular the theory of sifted-flatness comprises the following axioms:

SF1: For every pair of objects C', C'',

$$\forall x'\colon C' \ \ \forall x''\colon C'' \ \bigvee_{C} \ \bigvee_{\substack{u'\colon C\to C' \\ u''\colon C\to C''}} \ \exists x\colon C \ \ (u'(x) = x' \wedge u''(x) = x'')$$

Interpreting this formula relative to a site structure $(\mathcal{E}, j)$ means that it is satisfied at every stage $T \in \mathcal{E}$. In more plain terms this says that, whenever we have a diagram

$$\begin{tikzcd} & T \arrow[dl, "x'"'] \arrow[dr, "x''"] & \\ FC' & & FC'', \end{tikzcd}$$

there exists a cover $\{t_\alpha\colon T_\alpha \to T \mid \alpha \in A\}$ of T such that for each $\alpha \in A$ there exists an object C_a , morphisms $u : C_a \to C'$, $v : C_a \to C''$ and a morphism (generalized element) $x_\alpha : T_\alpha \to FC_a$, such the following diagram is commutative

$$\begin{tikzcd} T_\alpha \arrow[r, "t_\alpha"] \arrow[dr, "x_\alpha"] & T \arrow[ddl, "x'"'] \arrow[ddr, "x''"] & \\ & FC_a \arrow[dl, "Fu"] \arrow[dr, "Fv"'] & \\ FC' & & FC'' \end{tikzcd}$$

i.e

$$Fu \circ x_\alpha = x' \circ t_\alpha$$
$$Fv \circ x_\alpha = x'' \circ t_\alpha$$

SF2: For every diagram of the form

$$\begin{tikzcd} & Z' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ A & & B \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

$\forall z : Z\, \forall z' : Z'\, ((f(z) = f'(z') \wedge g(z) = g'(z')) \rightarrow \bigvee_{\mathcal{Z}(Z,Z')} \exists z_1 : Z_1\ \exists z_2 : Z_2 ... \exists z_n : Z_n (d_1^0(z_1) = z \wedge d_1^1(z_1) = z_2 \wedge ... \wedge d_n^0(z_n) = z_{n-1} \wedge d_n^1(z_n) = z'$

Here the disjunction is taken over the set $\mathcal{Z}(Z, Z')$ of all zig-zags connecting Z and Z' in the category $(A, B) \downarrow \mathcal{C}^{op}$ of all (A, B)-spans. Again, when stated in elementary terms, this condition takes the form that for any pair of arrows $z : T \to FZ$, $z' : T \to FZ'$, such that $Ff \cdot z = Ff' \cdot z'$ and $Fg \cdot z = Fg' \cdot z'$, there exists a j-cover $\{t_\alpha : T_\alpha \to T\}$, and, for each α, a zig-zag in the category $(A, B) \downarrow \mathcal{C}^{op}$ of all (A, B)-spans and arrows $z_{\alpha,i} : T_\alpha \to FZ_i$, $i = 1, ..., n$, to the images under F of the vertices Z_i of the zig-zag, satisfying the obvious commutativities.

2.9 Remark. Notice that when interpreting the above axioms with respect to various Grothendieck topologies, their validity in a site $(\mathcal{E}, j)$ implies their validity in any $(\mathcal{E}, j')$, where $j \leq j'$, i.e there are more j'-coverings. In particular, validity in the "absolute sense" of the previous paragraph (coarsest topology) implies validity with respect to any topology.

The above considerations give the correct notion of sifted-flatness and resolve the "contradiction" mentioned above: One just has to interpret the conditions defining sifted-flatness with respect, not to the coarsest topology, but with respect to a suitable subcanonical topology on **Cat**.

Still though certain complications may arise. For example, in the following situation

$$\begin{tikzcd}
\mathbf{BAlg_f} \arrow[rr, "y"] \arrow[dr, "i"'] & & {[\mathbf{BAlg_f}^{op}, \mathbf{Set}]} \arrow[dl, shift left, "Lan_y i"] \\
& \mathbf{BAlg} \arrow[ur, shift left, "S"] &
\end{tikzcd}$$

where $i : \mathbf{BAlg_f} \to \mathbf{BAlg}$ denotes the inclusion of finite Boolean algebras into all Boolean algebras, i is sifted flat (even with respect to the coarsest structure) because it preserves products. Also i is dense, hence **BAlg** is a reflective subcategory of $[\mathbf{BAlg_f}^{op}, \mathbf{Set}]$. At the same time $\mathrm{Lan}_y i$ does not preserve finite products. In such a case we would have as a conclusion that **BAlg** is cartesian closed (see [9], A4.3.1). Clearly this is not the case. Hence some extra condition is required, that would guarantee the preservation of finite products by the left Kan extension.

3 Postulated colimits

Whenever we have a diagram $D : \mathcal{D} \to \mathbf{Set}$, the colimit of it is given as a quotient of the coproduct of all the values of D divided by an equivalence relation. This description allows us, whenever dealing with colimits in **Set**, to argue about its elements in such a way that

1. An element of the colimit can be represented by an element of some component.

2. If an element of the colimit has two such representations, then there exists a zig-zag connecting the two sets, where the representations live, and a sequence of elements satisfying certain compatibility equations.

Anders Kock ([12]) realized that this behaviour of colimits in **Set** is the key ingredient that allows us to deduce preservation-of-finite-limit properties by the Kan extension of a functor, out of properties of the category of elements of the functor. He exploited the fact that these are properties expressible by geometric sentences in a suitable language and hence are soundly interpretable in sites and deduced preservation-of-finite-limit properties by Kan extensions of functor into sites. More precisely, given a diagram $D : \mathcal{D} \to \mathcal{E}$, where $(\mathcal{E}, j)$ is a site and a cocone $\{\mathrm{incl}_d : D(d) \to L | \, d \in \mathcal{D}\}$, we say that the cocone is j-postulated if it satisfies the following two conditions, which are written in the internal language of $\mathcal{E}$:

PC1: $\forall x : L \bigvee_{d \in \mathcal{D}} \exists y : Dd \;\; (\mathrm{incl}_d(y) = x)$ and

PC2: $\forall x : Dd \; \forall y : Dd' \;\; (\mathrm{incl}_d(x) = \mathrm{incl}_{d'}(y) \to \bigvee_{\mathcal{Z}(d,d')} \exists z_1 : Dd_1 ... \; \exists z_n : Dd_n \; (D\delta_{1,0}(z_1) = x \; \wedge D\delta_{1,1}(z_1) = D\delta_{2,0}(z_2) \; \wedge ... \wedge \; D\delta_{n,1}(z_n) = y)),$

where $\mathcal{Z}(d, d')$ stands for the set (class) of zig-zags from to d to d', as in

$$\begin{tikzcd}
& d_1 \arrow[dl, "\delta_{1,0}"'] \arrow[dr, "\delta_{1,1}"] & & & d_n \arrow[dl, "\delta_{n,0}"'] \arrow[dr, "\delta_{n,1}"] & \\
d & & \cdot \arrow[r, dotted, no head] & \cdot & & d'
\end{tikzcd}$$

Again, the interpretation of these conditions in a site $(\mathcal{E}, j)$ can be stated in an elementary way. The first one says that, for all $T \in \mathcal{E}$, all $x: T \to L$, there is a j-cover $\{t_\alpha: T_\alpha \to T\}$ and, for all α, arrows $y_{\alpha,d} : T_\alpha \to D_d$ to some D_d, such that $x \cdot t_\alpha = \mathrm{incl}_d \cdot y_{\alpha,d}$. The second condition says that, for all $T \in \mathcal{E}$ and $x: T \to Dd$, $y: T \to Dd'$, if $\mathrm{incl}_d \cdot x = \mathrm{incl}_{d'} \cdot y$, then there is a j-cover $\{t_\alpha: T_\alpha \to T\}$ so that, for all α, there are $z_{\alpha,i} : T_\alpha \to Dd_i$, $i = 1, ..., n$, to the images under D of the vertices of the zig-zag, satisfying the obvious commutativities.

3.1 Remark. (i) As noted in connection to the sifted-flatness conditions, validity of the above conditions in a site $(\mathcal{E}, j)$ implies their validity in any $(\mathcal{E}, j')$, where $j \leq j'$

(ii) One way to restate condition PC1 is that the colimit inclusions $\mathrm{incl}_d : Dd \to \mathrm{colim}\, D$ form a j-covering family (because, for the identity arrow of $\mathrm{colim}\, D$, the existence of a cover of $\mathrm{colim}\, D$ with the prescribed property implies that the sieve generated by the colimit inclusions is larger than that covering, hence also covering).

(iii) As explained in [12] (remark following the proof of Proposition 1.1), ignoring any size issues that may arise, postulated colimits with respect to a subcanonical site $(\mathcal{E}, j)$ are those that are preserved by the Yoneda embedding $\mathcal{E} \hookrightarrow \mathrm{shv}(\mathcal{E}, j)$. Hence the colimit of a diagram $D: \mathcal{D} \to \mathcal{E}$ is postulated iff the colimit of the restriction along any final functor $J: \mathcal{D}' \to \mathcal{D}$ is postulated. Nevertheless, in the applications it turns out that it is crucial that certain diagrams have finite final sets of cocones, for finding suitable subcanonical topologies so that the required colimits are postulated.

(iv) A. Kock shows that if finite (all) colimits are postulated with respect to a subcanonical topology on $\mathcal{E}$ then $\mathcal{E}$ is a(n ∞-) pretopos. So one may wonder whether this is the case with the main examples of categories where realization functors take values, such as **Cat** or the category of Kelley spaces. For **Cat** it is well-known that it fails even to be regular ([5]). With Kelley spaces the situation is subtler, as Cagliari, Mantovani and Vitale ([6]) show that this category, while it is regular, it is not exact. But as will become evident in the next section, if we are interested in showing the preservation of finite products by geometric or categorical realization functors only specific colimits need to be postulated with respect to some subcanonical topology. Hence we turn to showing that various useful colimits (colimits used in the calculation of certain values of left Kan extensions) have this property.

3.2 Proposition. *There exists a subcanonical topology on the category $\mathcal{K}$ of Kelley spaces such that the families of colimit inclusions $\{\mathrm{incl}_i\colon K_i \to \operatorname{colim} K_i\}$ of finite diagrams of compact Hausdorff spaces are covering.*

Proof: Consider the Grothendieck topology generated by families of colimit inclusions $\{\mathrm{incl}_i\colon K_i \to \operatorname{colim} K_i\}$ of finite diagrams of compact Hausdorff spaces. Such families are not stable under pullback along a continuous map from an arbitrary Kelley space. Hence we close the family under such pullbacks, generating thus a Grothendieck pretopology on $\mathcal{K}$.

If $\{f_i\colon K_i \to Z\}$ is a compatible family of continuous maps, then we may define

$$f\colon \operatorname{colim}_i K_i \to Z$$

by $f(x) = f_i(x_i)$, since every $x \in \operatorname{colim}_i K_i$ is $x = \mathrm{incl}_i(x_i)$, for some $x_i \in K_i$. This is a correct definition by the compatibility of the family. This f is continuous because, if $C \subseteq Z$ is closed, then $f^{-1}C = \bigcup_i \mathrm{incl}_i[f_i^{-1}C]$. Each $\mathrm{incl}_i[f_i^{-1}C]$ is closed because incl_i is closed, being a map between compact Hausdorff spaces, and the union is finite. Thus $f^{-1}C$ is closed. This proves that each representable functor $\mathcal{K}(-, Z)$ on Kelley spaces has the sheaf property with respect to the generating family of the pretopology.

Not only this, but $\mathcal{K}(-, Z)$ has the sheaf property with respect to a pullback $\{t_i\colon T_i \to T\}$ of a generating family along any map $T \to \operatorname{colim}_i K_i$: The extension map $T \to Z$ is defined as above because the underlying set of the pullback $T_i \times_T T_{i'}$ is the pullback of the underlying sets of the involved spaces (although the topology on it may not be inherited by the Tychonoff topology on $T_i \times T_{i'}$). The induced map is continuous by the same argument as above, because each $t_i\colon T_i \to T$ is closed: The maps $\mathrm{incl}_i\colon K_i \to \operatorname{colim}_i K_i = K$, being continuous maps between compact Hausdorff spaces, are stably closed and, moreover, the pullbacks $T_i \cong K_i \times_K T$ are computed as in the category of all topological spaces, since the K_i's are compact. ∎

3.3 Proposition. *Let $\mathcal{C}$ be a category with the property that for any finite diagram $D\colon \mathcal{D} \to \mathcal{C}$, the limit of the corresponding diagram of representables in $[\mathcal{C}^{op}, \mathbf{Set}]$ is isomorphic to a finite colimit of representables,*

$$\lim\ y \cdot D \ \cong\ \operatorname{colim}_k\, yP_k$$

If $F\colon \mathcal{C} \to \mathcal{K}$ is a functor to the category of Kelley spaces and the spaces FP_k are compact, then the colimit $\operatorname{colim}_k FP_k$ satisfies PC1 for the topology of Proposition 3.2.

Proof: This is an immediate consequence of Remark 3.1 (ii) ∎

3.4 Corollary. *If $\mathcal{C}$ is either $\mathbf{\Delta}$ or $\mathbb{G}$ (the basis of globular sets [4]), then the colimit used for the calculation of the left Kan extension of a limit $\lim\limits_{d\in\mathcal{D}} yD_d$ of representables, satisfies PC1 with respect to the subcanonical topology of Proposition 3.2.*

Proof: This is so because the limit of a finite diagram of representables on $\mathbf{\Delta}$ or $\mathbb{G}$ is a finite colimit of representables, as shown explicitly for $\mathbf{\Delta}$ in Proposition 2.7 and as can easily be seen for $\mathbb{G}$, invoking the arguments in the proof of Proposition 2.7. Moreover the representables are realized as simplices or closed balls, in the respective cases, hence are compact. ∎

3.5 Proposition. *Let $U\colon \mathbf{\Delta} \to \mathbf{Cat}$ be the obvious functor and $\mathrm{colim}_k\, U[p_k]$ the colimit arising from the description of a product of two representable simplicial sets as a finite colimit of representables. Then this colimit satisfies PC1 with respect to some subcanonical topology.*

Proof: Since the nerve functor $N\colon \mathbf{Cat} \to [\mathbf{\Delta}^{op}, \mathbf{Set}]$ is fully faithful and preserves pullbacks, we can easily see that there is a topology on $\mathbf{Cat}$ defined as follows: $\{\mathcal{C}_i \to \mathcal{C} \mid i \in I\}$ covers in $\mathbf{Cat}$ iff $\{N\mathcal{C}_i \to N\mathcal{C} \mid i \in I\}$ covers for the canonical topology on simplicial sets. It is also straightforward that this topology is subcanonical. Then the family $\{U[p_k] \to \mathrm{colim}_k\, U[p_k]\}$ is covering for this topology, essentially by Proposition 2.7, and the well-known fact that U is dense.

The fact that the colimit in question satisfies PC1 is again a consequence of Remark 3.1(ii). ∎

4 The main results

4.1 Proposition. *Let $F\colon \mathcal{C} \to \mathcal{E}$ be a functor from a small category $\mathcal{C}$ into a cocomplete, locally small, subcanonical site with finite products. Let also, for every pair of objects C_1, C_2 in $\mathcal{C}$, $\{P_k | k \in K\}$ be a diagram of cones for this pair of objects, so that the product of the two representables is a colimit of representables*

$$yC_1 \times yC_2 \cong \mathrm{colim}_k yP_k$$

and the colimit $\mathrm{colim}_k FP_k$ satisfies PC1 with respect to j. Then if F is j-sifted flat, the canonical morphism

$$f : \mathrm{colim}_k FP_k \to FC_1 \times FC_2$$

is an isomorphism

Proof: First we show that f is a monomorphism: Consider

$$\begin{tikzcd} T \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \mathrm{colim} FP_k \arrow[r, "f"] & FC_1 \times FC_2 \end{tikzcd}$$

with $f \cdot u = f \cdot v$

Using PC1 for the morphism $u : T \to \mathrm{colim} FP_k$ we have that there exists a cover $\{t_\alpha : T_\alpha \to T \mid \alpha \in A\}$ of T such that, for each $\alpha \in A$, there exists an index $k(\alpha) \in K$ and a morphism $u_\alpha : T_\alpha \to FP_{k(\alpha)}$, such that the following square commutes

$$\begin{tikzcd} T_\alpha \arrow[r, "t_\alpha"] \arrow[d, "u_\alpha"'] & T \arrow[d, "u"] \\ FP_{k(\alpha)} \arrow[r, "in_{k(\alpha)}"'] & \mathrm{colim} FP_k \end{tikzcd}$$

Similarily, for $v : T \to \mathrm{colim} FP_k$ we get, for each $\alpha \in A$, a commutative square

$$\begin{tikzcd} T_\alpha \arrow[r, "t_\alpha"] \arrow[d, "v_\alpha"'] & T \arrow[d, "v"] \\ FP_{k'(\alpha)} \arrow[r, "in_{k'(\alpha)}"'] & \mathrm{colim} FP_k \end{tikzcd}$$

We may assume that the same cover occurs in the two applications of PC1 above, since any two covers have a common refinement. The site is subcanonical so for the equality $u = v$, it is enough to show that $u \cdot t_\alpha = v \cdot t_\alpha$ for each $\alpha \in A$. Then, by the commutativity of the two squares above, we have to show $in_k \cdot u_\alpha = in_{k'} \cdot v_\alpha$.

Now consider the following diagram in $\mathcal{E}$ (where $p_i : FC_1 \times FC_2 \to FC_i$, $i = 1, 2$ are the projections from the product in $\mathcal{E}$):

$$\begin{tikzcd} & & FP_{k(\alpha)} \arrow[dl, "p_1 \cdot f \cdot in_{k(\alpha)}"] \arrow[dr, "p_2 \cdot f \cdot in_{k(\alpha)}"] & \\ T_\alpha \arrow[urr, "u_\alpha"] \arrow[drr, "v_\alpha"'] & FC_1 & & FC_2 \\ & & FP_{k'(\alpha)} \arrow[ul, "p_1 \cdot f \cdot in_{k'(\alpha)}"'] \arrow[ur, "p_2 \cdot f \cdot in_{k'(\alpha)}"'] & \end{tikzcd}$$

The equations

$$p_1 \cdot f \cdot in_{k(\alpha)} = Fp^1_{k(\alpha)}, \;\; p_1 \cdot f \cdot in_{k'(\alpha)} = Fp^1_{k'(\alpha)}, \;\; p_2 \cdot f \cdot in_{k(\alpha)} = Fp^2_{k(\alpha)}, \;\; p_2 \cdot f \cdot in_{k'(\alpha)} = Fp^2_{k'(\alpha)}$$

hold, where $in_{k(\alpha)} : FP_{k(\alpha)} \to \mathrm{colim} FP_k$ are the colimit injections and $p^i_k : P_k \to C_i$, $i = 1, 2$, are the projections from the cone P_k to C_1, C_2 in $\mathcal{C}$. The following equations hold

$$p_1 \cdot f \cdot in_{k'(\alpha)} \cdot v_a = p_1 \cdot f \cdot v \cdot t_\alpha = p_1 \cdot f \cdot u \cdot t_\alpha = p_1 \cdot f \cdot in_{k(\alpha)} \cdot u_\alpha$$

$$p_2 \cdot f \cdot in_{k'(\alpha)} \cdot v_\alpha = p_2 \cdot f \cdot v \cdot t_\alpha = p_2 \cdot f \cdot u \cdot t_\alpha = p_2 \cdot f \cdot in_{k(\alpha)} \cdot u_\alpha,$$

exploiting the commutativity of the two commutative squares above.

F is j- sifted flat, thus we can apply SF2: We have that, for all α there exists a cover of T_α, say $\{t_{\alpha\xi} : T_{\alpha\xi} \to T_\alpha | \, \xi \in \Xi_\alpha\}$, such that for every $\xi \in \Xi_\alpha$ there exists a zig-zag in $(C_1, C_2) \downarrow \mathcal{C}^{op}$ connecting $C_1 \xleftarrow{p^1_{k(\alpha)}} P_{k(\alpha)} \xrightarrow{p^2_{k(\alpha)}} C_2$ and $C_1 \xleftarrow{p^1_{k'(\alpha)}} P_{k'(\alpha)} \xrightarrow{p^2_{k'(\alpha)}} C_2$ as indicated in the diagram bellow (which, for the economy of the presentation we depict as one of length three):

$$\begin{tikzcd}
 & P_{k(\alpha)} \arrow[ddl, "p^2_{k(\alpha)}"'] \arrow[ddr, "p^1_{k(\alpha)}"] & \\
 & P_{k_3(\alpha)} \arrow[u, "d^1_3"] \arrow[d, "d^0_3"] \arrow[dl, "p^1_{k_3(\alpha)}"] \arrow[dr, "p^2_{k_3(\alpha)}"'] & \\
C_1 & P_{k_2(\alpha)} \arrow[l, "p^1_{k_2(\alpha)}"'] \arrow[r, "p^2_{k_2(\alpha)}"'] & C_2 \\
 & P_{k_1(\alpha)} \arrow[u, "d^1_1"] \arrow[d, "d^0_1"] \arrow[ul, "p^1_{k_1(\alpha)}"'] \arrow[ur, "p^2_{k_1(\alpha)}"] & \\
 & P_{k'(\alpha)} \arrow[uul, "p^1_{k'(\alpha)}"] \arrow[uur, "p^1_{k'(\alpha)}"'] &
\end{tikzcd}$$

Without loss of generality we took the components of the zig-zag to be from the final family of cones P_k. The reason is that each component of the zig-zag constitutes a cone for the discrete diagram in $\mathcal{C}$ therefore they factorize through a P_k.

Furthermore, condition SF2 gives, for every $\xi \in \Xi_\alpha$, generalized elements

$$\begin{tikzcd}
 & FP_{k_1(\alpha)} \\
T_{\alpha\xi} \arrow[ur, "x_1"] \arrow[r, "x_2"] \arrow[dr, "x_3"'] & FP_{k_2(\alpha)} \\
 & FP_{k_3(\alpha)}
\end{tikzcd}$$

such that the following diagram is commutative

$$\begin{tikzcd}
 & T_\alpha \arrow[d, "u_\alpha"] & \\
 & FP_{k(\alpha)} \arrow[ddr, "<Fp^1_{k(\alpha)}, Fp^2_{k(\alpha)}>"] & \\
 & FP_{k_3(\alpha)} \arrow[u, "Fd_3^1"'] \arrow[d, "Fd_3^0"'] \arrow[dr] & \\
T_{\alpha\xi} \arrow[uuur, "t_{\alpha\xi}"] \arrow[ur, "x_3"] \arrow[r, "x_2"'] \arrow[dr, "x_1"'] \arrow[dddr, "t_{\alpha\xi}"] & FP_{k_2(\alpha)} \arrow[r] & FC_1 \times FC_2 \\
 & FP_{k_1(\alpha)} \arrow[u, "Fd_1^1"] \arrow[d, "Fd_1^0"'] \arrow[ur] & \\
 & FP_{k'(\alpha)} \arrow[uur, "<Fp^1_{k'(\alpha)}, Fp^2_{k'(\alpha)}>"'] & \\
 & T_\alpha \arrow[u, "v_\alpha"'] &
\end{tikzcd}$$

i.e

$$Fd_1^0 \cdot x_1 = v_\alpha \cdot t_{\alpha\xi}, \quad Fd_1^1 \cdot x_1 = x_2, \quad Fd_3^0 \cdot x_3 = x_2, \quad Fd_3^1 \cdot x_3 = u_\alpha \cdot t_{\alpha\xi}$$

Also from the fact that we have a colimiting cocone we have the commutativity of the following diagram:

$$\begin{tikzcd}
 & FP_{k_1(\alpha)} \arrow[dl, "Fd_{1,0}"'] \arrow[dr, "Fd_{1,1}"] \arrow[ddr, "in_{k_1(\alpha)}"] & & FP_{k_3(\alpha)} \arrow[dl, "Fd_{3,0}"'] \arrow[dr, "Fd_{3,1}"] \arrow[ddl, "in_{k_3(\alpha)}"] & \\
FP_{k'(\alpha)} \arrow[drr, "in_{k'(\alpha)}"'] & & FP_{k_2(\alpha)} \arrow[d, "in_{k_2(\alpha)}"'] & & FP_{k(\alpha)} \arrow[dll, "in_{k(\alpha)}"] \\
 & & \mathrm{colim} FP_k & &
\end{tikzcd}$$

Combining all the above we have

$$\begin{aligned}
in_{k(\alpha)} \cdot u_\alpha \cdot t_{\alpha\xi} &= in_{k(\alpha)} \cdot Fd_3^1 \cdot x_3 \\
&= in_{k_3(\alpha)} \cdot x_3 \\
&= in_{k_2(\alpha)} \cdot Fd_3^0 \cdot x_3 \\
&= in_{k_2(\alpha)} \cdot x_2 \\
&= in_{k_2(\alpha)} \cdot Fd_1^1 \cdot x_1 \\
&= in_{k_1(\alpha)} \cdot x_1
\end{aligned}$$

$$
\begin{aligned}
&= in_{k'(\alpha)} \cdot Fd_1^0 \cdot x_1 \\
&= in_{k'(\alpha)} \cdot v_\alpha \cdot t_{\alpha\xi}
\end{aligned}
$$

so

$$in_{k(\alpha)} \cdot u_\alpha = in_{k'(\alpha)} \cdot v_\alpha$$

because they become equal when they are restricted along a cover.

Next we show that f is a split epi. Let $x = \langle x_1, x_2 \rangle : T \to FC_1 \times FC_2$, be an arbitrary generalized element of the product. An application of SF1 gives us a cover $\{t_\alpha : T_\alpha \to T | \alpha \in A\}$ of T, such that for every $\alpha \in A$ there exists an object $P_{k(\alpha)}$ in the final family of cones (again there is no loss of generality) and a generalized element $x_\alpha : T_\alpha \to FP_{k(a)}$ so that the following diagram is commutative

$$
\begin{tikzcd}
T_\alpha \arrow[dr, "t_\alpha"] \arrow[ddr, "x_\alpha"'] & & \\
 & T \arrow[ddl, "x_1"'] \arrow[ddr, "x_2"] & \\
 & FP_{k(\alpha)} \arrow[dl, "Fp^1_{k(\alpha)}"] \arrow[dr, "Fp^2_{k(\alpha)}"'] & \\
FC_1 & & FC_2
\end{tikzcd}
$$

i.e, $Fp^1_{k(\alpha)} \cdot x_\alpha = x_1 \cdot t_\alpha$ and $Fp^2_{k(\alpha)} \cdot x_\alpha = x_2 \cdot t_\alpha$ We will show that the family $\{in_{k(\alpha)} \cdot x_\alpha | \, \alpha \in A\}$ of elements of $\mathcal{E}(-, \mathrm{colim} FP_k)$ is a matching family for the cover $\{t_\alpha : T_\alpha \to T | \alpha \in A\}$. For this, consider the diagram

$$
\begin{tikzcd}
 & T_\alpha \arrow[r, "x_\alpha"] \arrow[dr, "t_\alpha"] & FP_{k(\alpha)} \arrow[dr, "in_{k(\alpha)}"] & \\
T_\alpha \times_T T_{\alpha'} \arrow[ur, "\pi^1_{\alpha\alpha'}"] \arrow[dr, "\pi^2_{\alpha\alpha'}"'] & & T & \mathrm{colim} FP_k \\
 & T_{\alpha'} \arrow[r, "x_{\alpha'}"'] \arrow[ur, "t_{\alpha'}"'] & FP_{k(\alpha')} \arrow[ur, "in_{k(\alpha')}"'] &
\end{tikzcd}
\qquad (*)
$$

for arbitary $\alpha, \alpha' \in A$.

We need to show that the outer diagram in the above is commutative. To that end, we have that the upper road followed by $p_1 \cdot f$, gives the same result as the down road, when followed by the same composite.

$$
\begin{tikzcd}
 & & FC_1 \\
\mathrm{colim} FP_k \arrow[r, "f"] & FC_1 \times FC_2 \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\
 & & FC_2
\end{tikzcd}
$$

Indeed:

$$
\begin{aligned}
p_1 \cdot f \cdot in_{k(\alpha)} \cdot x_\alpha \cdot \pi^1_{\alpha\alpha'} &= p_1 \cdot f_{k(\alpha)} \cdot x_a \cdot \pi^1_{\alpha\alpha'} \\
&= Fp^1_{k(\alpha)} \cdot x_\alpha \cdot \pi^1_{\alpha\alpha'} \\
&= x_1 \cdot t_\alpha \cdot \pi^1_{\alpha\alpha'} \\
&= x_1 \cdot t_{\alpha'} \cdot \pi^2_{\alpha\alpha'} \\
&= Fp^1_{k(\alpha')} \cdot x_{\alpha'} \cdot \pi^2_{\alpha\alpha'} \\
&= p_1 \cdot f_{k(\alpha')} \cdot x_{\alpha'} \cdot \pi^2_{\alpha\alpha'} \\
&= p_1 \cdot f \cdot in_{k(\alpha')} \cdot x_{\alpha'} \cdot \pi^2_{\alpha\alpha'},
\end{aligned}
$$

where $f_{k(\alpha)}$ is the unique factorization in $\mathcal{E}$ of the cone $FP_{k(\alpha)}$ through $FC_1 \times FC_2$.

Similarily the up and the down road gives the same result if we compose with $p_2 \cdot f$, i.e,

$$p_2 \cdot f \cdot in_{k(\alpha)} \cdot x_a \cdot \pi^1_{\alpha\alpha'} = p_2 \cdot f \cdot in_{k(\alpha')} \cdot x_{\alpha'} \cdot \pi^2_{\alpha\alpha'}.$$

From the above equations and the universal property of the product we have that

$$f \cdot in_{k(\alpha)} \cdot x_\alpha \cdot \pi^1_{\alpha\alpha'} = f \cdot in_{k(\alpha')} \cdot x_{\alpha'} \cdot \pi^2_{\alpha\alpha'}.$$

and finally since f is a monomorphism we have the desired equation

$$in_{k(\alpha)} \cdot x_\alpha \cdot \pi^1_{\alpha\alpha'} = in_{k(\alpha')} \cdot x_{\alpha'} \cdot \pi^2_{\alpha\alpha'}.$$

This shows that $\{in_{k(\alpha)} \cdot x_\alpha |\, \alpha \in A\}$ is matching family for the sheaf $\mathcal{E}(-, \mathrm{colim} FP_k)$ with respect to the specified cover. The sheaf property of $\mathcal{E}(-, \mathrm{colim} FP_k)$ gives a unique arrow $r : T \to \mathrm{colim} FP_k$ such that for every $\alpha \in A$, $r \cdot t_\alpha = in_{k(\alpha)} \cdot x_\alpha$

$$
\begin{tikzcd}
T_\alpha \arrow[rr, "x_\alpha"] \arrow[dr, "t_\alpha"] & & FP_{k(\alpha)} \arrow[dr, "in_{k(\alpha)}"] & \\
 & T \arrow[rr, dotted, "r"] & & \mathrm{colim} FP_k
\end{tikzcd}
$$

Now we have:

$$\begin{array}{rcl} p_1 \cdot f \cdot r \cdot t_\alpha & = & p_1 \cdot f \cdot in_{k(\alpha)} \cdot x_\alpha \\ & = & p_1 \cdot f_{k(\alpha)} \cdot x_\alpha \\ & = & Fp^1_{k(\alpha)} \cdot x_\alpha \\ & = & x_1 \cdot t_\alpha \end{array}$$

for every $\alpha \in A$, hence $p_1 \cdot f \cdot r = x_1$ since the site is subcanonical. By a similar argument $p_2 \cdot f \cdot r = x_2$ and in conclusion we have the desired factorization

$$f \cdot r = \langle x_1, x_2 \rangle = x \qquad \blacksquare$$

4.2 Remark. In the proof of the Proposition we made no use of PC2

4.3 Corollary. *Let $F{:}\,\mathcal{C} \to \mathcal{E}$ be a functor from a small category $\mathcal{C}$ into a cocomplete, locally small, subcanonical site with finite products which fullfils the assumptions of the Proposition 4.1, i.e the colimit $\mathrm{colim} FP_k$, entering the calculation of $\mathrm{Lan}_y F$ of a product of two representables, satisfies PC1 with respect to j and F is j-sifted-flat. Then its left Kan extension along the Yoneda embedding y preserves finite products of representables. Moreover, if $\mathcal{E}$ is cartesian closed, then the left Kan extension preserves finite products.*

Proof: Concerning the first claim we have

$$\begin{array}{rcl} \mathrm{Lan}_y F(yC_1 \times yC_2) & \cong & \mathrm{Lan}_y F(\mathrm{colim}_k yP_k) \\ & \cong & \mathrm{colim}_k \mathrm{Lan}_y F(yP_k) \\ & \cong & \mathrm{colim}_k FP_k \\ & \cong & FC_1 \times FC_2 \\ & \cong & \mathrm{Lan}_y F(yC_1) \times \mathrm{Lan}_y F(yC_2) \end{array}$$

Then, expressing an object in $[\mathcal{C}^{op}, \mathbf{Set}]$ as a colimit of representables, using the commutation of binary products with colimits in that category as well as in $\mathcal{E}$ and the preservation of colimits by $\mathrm{Lan}_y F$, we conclude that preserves $\mathrm{Lan}_y F$ finite products. $\blacksquare$

4.4 Proposition. *Let $F{:}\,\mathcal{C} \to \mathcal{E}$ be a functor from a small category $\mathcal{C}$ into a cocomplete, locally small, subcanonical site with finite products. Let also, for every pair of objects C_1, C_2 in $\mathcal{C}$, $\{P_k | k \in K\}$ be a diagram of cones for this pair of objects, so that the product of the two representables is a colimit of representables*

$$yC_1 \times yC_2 \cong \mathrm{colim} yP_k$$

and the colimit $\mathrm{colim} FP_k$ is postulated with respect to j. Assume that the morphism $f : \mathrm{colim} FP_k \to FC_1 \times FC_2$ is an isomorphism. Then F is j-sifted flat. In particular this is the case when $\mathrm{Lan}_y F$ preserves finite products.

Proof: We denote with $f^{-1} : FC_1 \times FC_2 \to \mathrm{colim} FP_k$ the inverse of f.

Consider

$$\begin{tikzcd}
 & T \arrow[dl, "x_1"'] \arrow[dr, "x_2"] & \\
FC_1 & & FC_2
\end{tikzcd}$$

If x stands for the morphism $< x_1, x_2 >: T \to FC_1 \times FC_2$, we apply PC1 for the morphism $f^{-1} \cdot x : T \to \mathrm{colim} FP_k$. So we have a cover $\{T_\alpha \to T | \alpha \in A\}$ of T such that, for each $\alpha \in A$, there exist $P_{k(\alpha)}$ and $x_\alpha : T_\alpha \to FP_{k(\alpha)}$ making commutative the following square

$$\begin{tikzcd}
T_\alpha \arrow[r, "t_\alpha"] \arrow[d, "x_\alpha"'] & T \arrow[d, "f^{-1} \cdot x"] \\
FP_{k(\alpha)} \arrow[r, "in_{k(\alpha)}"'] & \mathrm{colim} FP_k
\end{tikzcd}$$

So we have the following diagram

$$\begin{tikzcd}
T_\alpha \arrow[dr, "t_\alpha"] \arrow[ddr, dotted, "x_\alpha"'] & & \\
 & T \arrow[ddl, "x_1"'] \arrow[ddr, "x_2"] & \\
 & FP_{k(\alpha)} \arrow[dl, "Fp^1_{k(\alpha)}"] \arrow[dr, "Fp^2_{k(\alpha)}"'] & \\
FC_1 & & FC_2
\end{tikzcd}$$

which is commutative.

Indeed

$$\begin{aligned}
Fp^1_{k(\alpha)} \cdot x_\alpha &= p_1 \cdot f_{k(\alpha)} \cdot x_\alpha \\
&= p_1 \cdot f \cdot in_{k(a)} \cdot x_a \\
&= p_1 \cdot f \cdot f^{-1} \cdot x \cdot t_\alpha \\
&= p_1 \cdot x \cdot t_\alpha \\
&= x_1 \cdot t_\alpha
\end{aligned}$$

and similarily

$$Fp^2_{k(\alpha)} \cdot x_\alpha = x_2 \cdot t_\alpha$$

So the first condition of sifted-flatness with respect to j holds.

Next, for arbitary C_1, C_2 in $\mathcal{C}$ consider the diagram

$$\begin{tikzcd}
& T \arrow[d, "z'"'] & \\
& FP_{k'} \arrow[dl, "Fp^1_{k'}"'] \arrow[dr, "Fp^2_{k'}"] & \\
FC_1 & & FC_2 \\
& FP_k \arrow[ul, "Fp^1_k"] \arrow[ur, "Fp^2_k"'] & \\
& T \arrow[u, "z"] &
\end{tikzcd}$$

where

$$Fp^1_k \cdot z = Fp^1_{k'} \cdot z' \quad \text{and} \quad Fp^2_k \cdot z = Fp^2_{k'} \cdot z' \tag{1}$$

(In the above diagram we can take without loss of generality the P_k's to be in the final family of cones for C_1, C_2.) Then

$$\begin{aligned}
p_1 \cdot f \cdot in_k \cdot z &= p_1 \cdot f_k \cdot z \\
&= Fp^1_k \cdot z \\
&= Fp^1_{k'} \cdot z' \\
&= p_1 \cdot f_{k'} \cdot z' \\
&= p_1 \cdot f \cdot in_{k'} \cdot z'
\end{aligned}$$

and similarily

$$p_2 \cdot f \cdot in_k \cdot z = p_2 \cdot f \cdot in_{k'} \cdot z'$$

i.e,

$$f \cdot in_k \cdot z = f \cdot in_{k'} \cdot z'$$

and since f is a mono we have that

$$in_k \cdot z = in_{k'} \cdot z'$$

By an application of PC2 for the colimit cocone $\{in_k : FP_k \to \text{colim} FP_k\}$ we have that there exists a cover $\{T_\alpha \to T | \alpha \in A\}$ such that for every $\alpha \in A$ we have a zig-zag (in $\mathcal{C}$ consisting of objects in the final family connecting P_k and $P_{k'}$

$$\begin{tikzcd}
& P_{k_1} \arrow[dl, "d_{1,0}"'] \arrow[dr, "d_{1,1}"] & & & P_{k_n} \arrow[dl, "d_{n,0}"'] \arrow[dr, "d_{n,1}"] & \\
P_k & & \cdot \arrow[r, dotted, no head] & \cdot & & P_{k'}
\end{tikzcd}$$

and generalized elements $x_1: T_\alpha \to FP_{k_1}$ ... $x_n: T_\alpha \to FP_{k_n}$ such that

$$Fd_1^0 \cdot x_1 = z \cdot t_\alpha, \;\; Fd_1^1 \cdot x_1 = x_2, \;\; Fd_3^0 \cdot x_3 = x_2, \;\; Fd_3^1 \cdot x_3 = x_4, ..., Fd_n^1 \cdot x_n = z' \cdot t_\alpha$$

Hence the second condition for sifted-flatness with respect to j holds. ■

4.5 Corollary. *Let $F: \mathcal{C} \to \mathcal{E}$ be a functor from a small category $\mathcal{C}$ into a Grothendieck topos. Then F is sifted flat in the internal logic of the topos (i.e with respect to the canonical topology of the topos) if and only if* $\mathrm{Lan}_y F$ *preserves finite products.*

Proof: Consider the topos equipped with its canonical topology. Then all colimits are postulated with respect to it ([12], Proposition 2.1). ■

4.6 Remark. Using the above for $\mathcal{E} = \mathbf{Set}$ and F the composite $\mathcal{C}^{op} \to \mathbf{1} \to \mathbf{Set}$ (where the second part of the composite chooses a singleton) and $\mathcal{C}$ is sifted, we get that $\mathrm{Lan}_y F$, which is just colimit formation for $\mathcal{C}$-diagrams, preserves finite products. Hence sifted colimits commute with finite products in sets, as anticipated in Remark 2.2.

4.7 Corollary. *Let $\mathcal{S}$ be a base topos, $\mathcal{C}$ an internal category in it that is dually sifted and j a topology such that every cover R of an object $C \in \mathcal{C}$ in it is connected, i.e it is connected as a subcategory of the category of elements of C. Then the induced geometric morphism* $\mathrm{Shv}(\mathcal{C}, j) \to \mathcal{S}$ *has the property that its inverse image has a further left adjoint that preserves finite products.*

Proof: There is a geometric morphism $[\mathcal{C}^{op}, \mathcal{S}] \to \mathcal{S}$ which has a further left adjoint, given by internal left Kan extension along $\mathcal{C}^{op} \to \mathbf{1}^{op}$. When $\mathcal{C}$ is dually sifted, that Kan extension preserves finite products. If moreover every j-cover is connected, so that every constant presheaf is a j-sheaf, then that extra left adjoint restricts to one $\mathrm{Shv}(\mathcal{C}, j) \to \mathcal{S}$, which also preserves finite products. ■

References

[1] J. Adamek, J. Rosicky, On sifted colimits and generalized varietes, *Theory and Applications of Categories*, Vol. 8, No. 3, 33-53 (2001)

[2] C. Barwick, D. M. Kan, Relative categories: Another model for the homotopy theory of homotopy theories, arXiv 1011.1691v1, (2010)

[3] T. Beke, Theories of presheaf type, *Journal of Symbolic Logic* 69 (3), 923-934 (2004)

[4] C. Berger, A cellular nerve for higher categories, *Advances in Mathematics* 169, 118-175 (2002)

[5] F. Borceux, *Handbook of Categorical Algebra 1, 2*, Encyclopedia of Mathematics and its Applications, Volumes 50-51., Cambridge University Press (1994)

[6] F. Cagliari, S. Mantovani, E. M. Vitale, Regularity of the category of Kelley spaces, *Appl. Categ. Structures* 3, 357–361 (1995)

[7] Nicholas Duncan, Gros and Petit Toposes, Talk given at PSSL88, Cambridge, available as http://www.cheng.staff.shef.ac.uk/pssl88/pssl88-duncan.pdf

[8] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag Berlin-Heidelberg (1967)

[9] P. T. Johnstone, *Sketches of an Elephant: a topos theory compendium: vol.1 and vol.2*, Oxford Logic Guides 43 and 44. Clarendon Press, Oxford (2002)

[10] P. Karazeris, J. Rosický, J. Velebil, Completeness of cocompletions, *Journal of Pure and Applied Algebra*, 169, 229-250 (2005)

[11] Katsov, On diagrams and flatness of functors, *Journal of Pure and Applied Algebra*, 154, 247-256 (2000).

[12] A. Kock, Postulated colimits and left exactness of Kan extensions, Aarhus Preprint 1989/90 no. 9, Retyped in TeX in the fall of 2003, available at http://home.imf.au.dk/kock/.

[13] A. Kock, *Synthetic Differential Geometry*, London Math. Society Lecture NotesSeries No 333, Cambridge University Press, second edition (2006)

[14] S. MacLane, L. Moerdijk, *Sheaves in Geometry and Logic: a first introduction to Topos Theory*, Springer-Verlag New York, (1992)

[15] E. Riehl, A model structure for quasicategories, (2010) available as http://math.uchicago.edu/ eriehl/topic.pdf.