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A DIMENSION-REDUCING METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS IN $\mathbb{R}^n$

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A method for the numerical solution of systems of nonlinear algebraic and/or transcendental equations in $\mathbb{R}^n$ is presented. This method reduces the dimensionality of the system in such a way that it can lead to an iterative approximate formula for the computation of $n-1$ components of the solution, while the remaining component of the solution is evaluated separately using the final approximations of the other components. This $(n-1)$-dimensional iterative formula generates a sequence of points in $\mathbb{R}^{n-1}$ which converges quadratically to $n-1$ components of the solution. Moreover, it does not require a good initial guess for one component of the solution and it does not directly perform function evaluations, thus it can be applied to problems with imprecise function values. A proof of convergence is given and numerical applications are presented.

KEY WORDS: Implicit function theorem, Newton's method, reduction to one-dimensional equations, nonlinear SOR, m-step SOR-Newton, imprecise function values, bisection method, systems of nonlinear equations, numerical solution, zeros, quadratic convergence.

C.R. CATEGORY: G.1.5.

1. INTRODUCTION

Suppose that $F=(f_1,\ldots,f_n):\mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n$ is a continuously differentiable mapping on an open neighborhood $\mathscr{D}^*\subset\mathscr{D}$ of a solution $x^*\in\mathscr{D}$ of the system of nonlinear equations

$$F(x)=\Theta^n=(0,0,\ldots,0). \tag{1.1}$$

There is a class of methods for the numerical solution of the above system which arise from iterative procedures used for systems of linear equation [7, 10–12, 14]. These methods use reduction to simpler one-dimensional nonlinear equations for the components $f_1, f_2, \ldots, f_n$ of F. The best-known method of this type is the *nonlinear successive overrelaxation* (*SOR*) method which solves the one-dimensional equation

$$f_i(x_1^{p+1},\ldots,x_{i-1}^{p+1},x_i,x_{i+1}^p,\ldots,x_n^p)=0, \tag{1.2}$$

*The work of the second author was done at the Department of Mathematics of Cornell University.

for x_i and then sets

$$x_i^{p+1}=x_i^p+\omega(x_i-x_i^p), \quad i=1,\ldots,n, \quad p=0,1,\ldots, \tag{1.3}$$

provided that $\omega\in(0,1]$. Independent of the value of ω the above process is called *SOR* process even though this nomenclature is sometimes reserved for the case $\omega>1$. Now, a large variety of combined methods can be constructed depending on the secondary iteration and the number of steps required for solving (1.2). Thus, for example, one can obtain the *exact nonlinear SOR* or *m-step SOR-Newton process* [11, 14] and so on. Now, if the Jacobian of F at the solution x^* of (1.1) is an $\mathscr{M}$-matrix [11] the iterates of the above processes will converge linearly to x^* provided that $\omega\in(0,1]$ [11].

It is well-known the Newton's method which starting with an initial guess x^0 for the attainment of an approximation of the solution x^* of (1.1) is given by

$$x^{p+1}=x^p-F'(x^p)^{-1}F(x^p), \quad p=0,1,\ldots. \tag{1.4}$$

Now, if the Jacobian $F'(x^*)$ is nonsingular and $F'(x)$ is Lipschitz continuous then the iterates (1.4) converge quadratically to x^* provided the initial guess x^0 is sufficiently close to x^*. The quadratic convergence of Newton's method is attractive. However, the method depends on a good initial approximation [3] and it requires in general n^2+n function evaluations per iteration besides the solution of an $n\times n$ linear system. Moreover, the behavior of Newton's method is problematic when $F'(x^*)$ is singular since in that case (1.4) does not converge quadratically and, in general, is not appropriate for approximations of x^* with a high accuracy. For this reason there are procedures [23, 24] which under some assumptions (such as rank $F'(x^*)=n-1$) can attain a highly accurate solution x^* by enlarging the system (1.1) to one which is at least $(2n+1)$-dimensional [23, 24]. Also, Newton's method remains problematic when the values of F cannot be accurately achieved. Of course, this problem is common to all iterative procedures which directly depend on function evaluations. To overcome it, one may resort to generalized bisection methods [2, 5, 6, 17–20, 22] since they only make use of the algebraic sign of the function involved in the equations. These methods, however do not generally attain a quadratic convergence.

In this paper, we derive and apply a new iterative procedure, for the numerical solution of systems of nonlinear algebraic and/or transcendental equations in $\mathbb{R}^n$, which incorporates the advantages of SOR and Newton algorithms. The new method, which in fact constitutes a generalization of a recent proposed method [4], is derived in such a way that it can maintain the advantages of this method. More specifically, although the method in [4] uses reduction to simpler one-dimensional nonlinear equations, it generates a quadratically converging sequence of points in $\mathbb{R}$ which converges to one component of the solution separately from the other component. Afterwards the second component is evaluated by one simple computation. Also, the method in [4] has the advantage that it does not require a good initial guess for both components of the solution and does not directly perform function evaluations, thus it can be applied to problems with imprecise function values. Moreover, it compares favourably with Newton's method when the Jacobian at the solution is singular, (without making any enlargement of the system), or when it is difficult to evaluate the function values accurately.

The generalized method is derived in Section 2 of this paper. In Section 3 we give a proof of its convergence and in Section 4 we illustrate it on a number of numerical applications.

2. DERIVATION OF THE METHOD

NOTATION 2.1. Throughout this paper $\mathbb{R}^n$ is the n-dimensional real space of column vectors x with components $x_1, x_2, \ldots, x_n$, $(y; z)$ represents the column vector with components $y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_k$, $\partial_i f(x)$ denotes the partial derivative of $f(x)$ with respect to the ith variable x_i, $\bar{\mathscr{A}}$ denotes the closure of the set $\mathscr{A}$ and $f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$ defines the mapping obtained by holding $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ fixed.

The following theorem and corollary will play a central role in the development of our analysis.

THEOREM 2.1. (Implicit Function Theorem). *Suppose that* $F = (f_1, \ldots, f_n) : \mathscr{D} \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ *is defined and continuously differentiable on an open neighborhood* $\mathscr{D}^0 \subset \mathscr{D}$ *of a point* $(x^0; y^0) = (x_1^0, \ldots, x_m^0, y_1^0, \ldots, y_n^0) \in \mathscr{D}$ *such that* $F(x^0; y^0) = \Theta^n$ *and that the Jacobian* $\partial(f_1, \ldots, f_n)/\partial(y_1, \ldots, y_n)$ *is nonsingular at* $(x^0; y^0)$. *Then there exist open neighborhoods* $\mathscr{A}_1 \subset \mathbb{R}^m$ *and* $\mathscr{A}_2 \subset \mathbb{R}^n$ *of* x^0 *and* y^0, *respectively, such that, for any* $x \in \bar{\mathscr{A}}_1$ *there is a unique system on n mappings* ϕ_i, $i = 1, \ldots, n$ *defined and continuous on* $\bar{\mathscr{A}}_1$ *such that* $y_i = \phi_i(x) \in \bar{\mathscr{A}}_2$ *for* $i = 1, \ldots, n$ *and* $f_i(x, \phi_1(x), \ldots, \phi_n(x)) = 0$ *for* $i = 1, \ldots, n$ *and any* $x \in \bar{\mathscr{A}}_1$. *Moreover the function* $\Phi = (\phi_1, \ldots, \phi_n)$ *is continuously differentiable in* $\mathscr{A}_1$ *and the Jacobian matrix* $\Phi'(x)$ *is equal to* $-B^{-1}C$, *where* C *(respectively* B*) is obtained by replacing* y_i *by* $\phi_i(x)$, $i = 1, \ldots, n$ *in the Jacobian matrix* $[\partial f_i/\partial x_k]$ *(respectively* $[\partial f_i/\partial y_j]$*).*

Proof See [1, 11].

A direct corollary of the above theorem is the following.

COROLLARY 2.1. *Suppose that* $f : \mathscr{D} \subset \mathbb{R}^n \to \mathbb{R}$ *is defined and continuously differentiable on an open neighborhood* $\mathscr{D}^0 \subset \mathscr{D}$ *of a point* $x^0 = (x_1^0, \ldots, x_n^0)$ *for which* $f(x^0) = 0$ *and* $\partial_n f(x^0) \neq 0$. *Then there exist open neighborhoods* $\mathscr{A}_1 \subset \mathbb{R}^{n-1}$ *and* $\mathscr{A}_2 \subset \mathbb{R}$ *of the points* $y^0 = (x_1^0, \ldots, x_{n-1}^0)$ *and* x_n^0 *respectively, such that, for any* $y = (x_1, \ldots, x_{n-1}) \in \bar{\mathscr{A}}_1$ *there is a unique mapping* ϕ *defined and continuous on* $\bar{\mathscr{A}}_1$ *such that* $x_n = \phi(y) \in \bar{\mathscr{A}}_2$ *and* $f(y; \phi(y)) = 0$ *for any* $y \in \bar{\mathscr{A}}_1$. *Moreover the mapping* $\phi : \mathscr{A}_1 \to \mathbb{R}$ *has continuous partial derivatives in* $\mathscr{A}_1$ *which are given by*

$$\partial_j \phi(y) = -\partial_j f(y; \phi(y))/\partial_n f(y; \phi(y)), \quad j = 1, \ldots, n-1. \tag{2.1}$$

Of course, relative corollaries can be obtained using any one of the components $x_1,\ldots,x_n$, for example x_i, instead of x_n and taking $y=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$.

Next, we shall implement the above results to derive a method for solving systems of nonlinear algebraic and/or transcendental equations in $\mathbb{R}^n$. To do this, assume that $F=(f_1,\ldots,f_n):\mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n$ is twice-continuously differentiable on an open neighborhood $\mathscr{D}^*\subset\mathscr{D}$ of a solution $x^*=(x_1^*,\ldots,x_n^*)\in\mathscr{D}$ of the system of nonlinear equations

$$F(x)=\Theta^n. \tag{2.2}$$

Our interest lies in obtaining an approximation of x^*. So, we consider the sets $\mathscr{B}_i$, $i=1,\ldots,n$ to be those connected components of $f_i^{-1}(0)$ containing x^* on which $\partial_n f_i\neq 0$, for $i=1,\ldots,n$ respectively. Next, we apply Corollary 2.1 for each one of the components f_i, $i=1,\ldots,n$ of F. So, according to the above corollary there exist open neighborhoods $\mathscr{A}_1^*\subset\mathbb{R}^{n-1}$ and $\mathscr{A}_{2,i}^*\subset\mathbb{R}$, $i=1,\ldots,n$ of the points $y^*=(x_1^*,\ldots,x_{n-1}^*)$ and x_n^* respectively, such that for any $y=(x_1,\ldots,x_{n-1})\in\bar{\mathscr{A}}_1^*$ there exist unique mappings ϕ_i defined and continuous in $\mathscr{A}_1^*$ such that

$$x_n=\phi_i(y)\in\bar{\mathscr{A}}_{2,i}^*,\quad i=1,\ldots,n, \tag{2.3}$$

and

$$f_i(y;\phi_i(y))=0,\quad i=1,\ldots,n. \tag{2.4}$$

Moreover there exist the partial derivatives $\partial_j\phi_i$, $j=1,\ldots,n-1$ in $\mathscr{A}_1^*$ for each ϕ_i, $i=1,\ldots,n$, they are continuous in $\bar{\mathscr{A}}_1^*$ and they are given by

$$\partial_j\phi_i(y)=-\partial_j f_i(y;\phi_i(y))/\partial_n f_i(y;\phi_i(y)),\quad i=1,\ldots,n,\quad j=1,\ldots,n-1. \tag{2.5}$$

Suppose now that $x^0=(x_1^0,\ldots,x_n^0)$ is an initial approximation of the solution x^* where $y^0=(x_1^0,\ldots,x_{n-1}^0)\in\bar{\mathscr{A}}_1^*$, then using Taylor's formula we can expand the $\phi_i(y)$, $i=1,\ldots,n$ about y^0 where $y=(x_1,\ldots,x_{n-1})$. So, we can obtain that

$$\phi_i(y)\simeq\phi_i(y^0)+\sum_{j=1}^{n-1}(x_j-x_j^0)\partial_j\phi_i(y^0),\quad i=1,\ldots,n. \tag{2.6}$$

Now, using the relationships (2.3) and (2.5) we form the following system of equations,

$$x_n=x_n^{0,i}-\sum_{j=1}^{n-1}(x_j-x_j^0)\,\partial_j f_i(y^0;x_n^{0,i})/\partial_n f_i(y^0;x_n^{0,i}),\quad i=1,\ldots,n, \tag{2.7}$$

where $x_n^{0,i}=\phi_i(y^0)$, $i=1,\ldots,n$ are the corresponding solutions of the one-dimensional equations of one unknown $f_i(x_1^0,\ldots,x_{n-1}^0,\cdot)=0$, $i=1,\ldots,n$.

Next, from the nth equation of the above set of equations we can obtain that

$$x_n = x_n^{0,n} - \sum_{j=1}^{n-1} (x_j - x_j^0)\, \partial_j f_n(y^0; x_n^{0,n}) / \partial_n f_n(y^0; x_n^{0,n}). \tag{2.8}$$

By substituting (2.8) in the Eqs. (2.7) we obtain the following system of $n-1$ linear equations

$$\sum_{j=1}^{n-1} (x_j - x_j^0)\,(\partial_j f_i(y^0; x_n^{0,i}) / \partial_n f_i(y^0; x_n^{0,i}) - \partial_j f_n(y^0; x_n^{0,n}) / \partial_n f_n(y^0; x_n^{0,n})) = x_n^{0,i} - x_n^{0,n}, \quad i = 1, \ldots, n-1, \tag{2.9}$$

which, in matrix form, becomes

$$A_0(y - y^0) = V_0, \tag{2.10}$$

where

$$\begin{aligned} A_0 &= [a_{ij}] = [\partial_j f_i(y^0; x_n^{0,i}) / \partial_n f_i(y^0; x_n^{0,i}) \\ &\qquad - \partial_j f_n(y^0; x_n^{0,n}) / \partial_n f_n(y^0; x_n^{0,n})], \quad i,j = 1, \ldots, n-1, \\ y &= [x_i], \quad y^0 = [x_i^0], \quad i = 1, \ldots, n-1, \\ V_0 &= [v_i] = [x_n^{0,i} - x_n^{0,n}], \quad i = 1, \ldots, n-1. \end{aligned} \tag{2.11}$$

Assuming that A_0 is nonsingular, the solution y of the linear system (2.10) gives a new approximation of the first $n-1$ components of the solution x^* of (2.1) and finally, by replacing y in (2.8) we can approximate the nth component of x^*. Thus in general we can obtain the following iterative scheme for the computation of the $n-1$ components of x^*

$$y^{p+1} = y^p + A_p^{-1} V_p, \quad p = 0, 1, \ldots, \tag{2.12}$$

where

$$\begin{aligned} y^p &= [x_i^p], \quad i = 1, \ldots, n-1, \\ A_p &= [a_{ij}] = [\partial_j f_i(y^p; x_n^{p,i}) / \partial_n f_i(y^p; x_n^{p,i}) \\ &\qquad - \partial_j f_n(y^p; x_n^{p,n}) / \partial_n f_n(y^p; x_n^{p,n})], \quad i,j = 1, \ldots, n-1, \\ V_p &= [v_i] = [x_n^{p,i} - x_n^{p,n}], \quad i = 1, \ldots, n-1. \end{aligned} \tag{2.13}$$

Finally, after a desired number of iterations of the above scheme, say $p = m$, using (2.8) we can approximate the nth component of x^* using the following relationship

$$x_n^{m+1} = x_n^{m,n} - \sum_{j=1}^{n-1} (x_j^{m+1} - x_j^m)\, \partial_j f_n(y^m; x_n^{m,n}) / \partial_n f_n(y^m; x_n^{m,n}). \tag{2.14}$$

Of course, relative procedures for obtaining x^* can be constructed by replacing x_n in Corollary 2.1 with any one of the components $x_1, \ldots, x_{n-1}$, for example x_i, and taking $y = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

We would like to mention here that the above process does not require the expressions ϕ_i but only the values $x_n^{p,i}$ which are given by the solution of the one-dimensional equations $f_i(x_1^p, \ldots, x_{n-1}^p, \cdot) = 0$. So, by holding $y^p = (x_1^p, \ldots, x_{n-1}^p)$ fixed we can solve the equations

$$f_i(y^p; r_i^p) = 0, \quad i = 1, \ldots, n, \tag{2.15}$$

for r_i^p in the interval $(\alpha, \alpha + \beta)$ with an accuracy δ. Of course, we can use any one of the well-known one-dimensional methods [11, 13, 14, 16] to solve the above equations. Here we shall use the one-dimensional bisection, (see [2, 15] for a discussion of its advantages), since frequently the steps β are long and also few significant digits are required for the computations of the roots of the Eqs. (2.15). A simplified version of the bisection method can be found in [17, 19–22]. For completeness, we shall give here a brief description of this method. Hence, to solve an equation of the form

$$\psi(t) = 0, \tag{2.16}$$

where $\psi: [\gamma_1, \gamma_2] \subset \mathbb{R} \to \mathbb{R}$ is continuous, a simplified version of the bisection method leads to the following iterative formula

$$t_{k+1} = t_k + \operatorname{sgn} \psi(t_0) \cdot \operatorname{sgn} \psi(t_k) \cdot h / 2^{k+1}, \quad k = 0, 1, \ldots, \tag{2.17}$$

with $t_0 = \gamma_1$ and $h = \gamma_2 - \gamma_1$ and where for any real number a,

$$\operatorname{sgn} a = \begin{cases} -1, & \text{if } a < 0, \\ 0, & \text{if } a = 0, \\ 1, & \text{if } a > 0. \end{cases} \tag{2.18}$$

Of course, (2.17) converges to a root $t^* \in (\gamma_1 - \gamma_2)$ if for some t_k, $k = 0, 1, \ldots$ holds that

$$\operatorname{sgn} \psi(t_0) \cdot \operatorname{sgn} \psi(t_k) = -1. \tag{2.19}$$

Also, the minimum number of iterations μ, that are required in obtaining an approximate root $\hat{t}$ such that $|\hat{t} - t^*| \leq \varepsilon$, for some $\varepsilon \in (0, 1)$ is given by

$$\mu = \lceil \log_2(h \cdot \varepsilon^{-1}) \rceil, \tag{2.20}$$

where the notation $\lceil v \rceil$ refers to the least integer that is not less than the real number v.

For a geometric interpretation of the new method and a corresponding illustration of the main differences between Newton's method and new method, we refer the interested reader to [4].

3. A PROOF OF CONVERGENCE

We shall give in this section a proof of the convergence of the new method described by the iterates (2.12) and the relationship (2.14). To this end the following theorem will be needed.

THEOREM 3.1. *Suppose that $F=(f_1,\dots,f_k)$: $\mathscr{E}\subset\mathbb{R}^k\to\mathbb{R}^k$ is twice-continuously differentiable on an open neighborhood $\mathscr{E}^*\subset\mathscr{E}$ of a point $x^*=(x_1^*,\dots,x_k^*)\in\mathscr{E}$ for which $F(x^*)=\Theta^k$ and $F'(x^*)$ nonsingular. Then the iterates x^p, $p=0,1,\dots$ of Newton's method*

$$x^{p+1}=x^p-F'(x^p)^{-1}F(x^p),\quad p=0,1,\dots, \tag{3.1}$$

will converge to x^* provided the initial guess x^0 is sufficiently close to x^*. Moreover the order of convergence will be two.

Proof See [9, 13, 16].

We note here that the condition that $F'(x)$ be Lipschitz continuous in $\mathscr{E}^*$, (which we assumed in Section 1), is ensured since the component functions f_i of F are all twice-continuously differentiable. We now proceed with the following convergence result.

THEOREM 3.2. *Suppose that $F=(f_1,\dots,f_n)$: $\mathscr{D}\subset\mathbb{R}^n\to\mathbb{R}^n$ is twice-continuously differentiable on an open neighborhood $\mathscr{D}^*\subset\mathscr{D}$ of a point $x^*=(x_1^*,\dots,x_n^*)\in\mathscr{D}$ for which $F(x^*)=\Theta^n$. Let $\mathscr{B}_i$, $i=1,\dots,n$ be those connected components of $f_i^{-1}(0)$, containing x^* on which $\partial_n f_i\neq 0$ for $i=1,\dots,n$ respectively. Then the iterates of (2.12) and the relationship (2.14) will converge to x^* provided the matrix A_* which is obtained from the matrix A_p of (2.12) at x^* is nonsingular and also provided the initial guess $y^0=(x_1^0,\dots,x_{n-1}^0)$ is sufficiently close to $y^*=(x_1^*,\dots,x_{n-1}^*)$. Moreover the iterates y^p, $p=0,1,\dots$ of (2.12) have order of convergence two.*

Proof Obviously, the iterates (2.12) can be written as follows

$$y^{p+1}=y^p-W_p^{-1}V_p,\quad p=0,1,\dots, \tag{3.2}$$

where

$$y^p=[x_i^p], \quad i=1,\dots,n-1,$$

$$\begin{aligned} W_p=[w_{ij}]=[&-\partial_j f_i(y^p;x_n^{p,i})/\partial_n f_i(y^p;x_n^{p,i}) \\ &-(-\partial_j f_n(y^p;x_n^{p,n})/\partial_n f_n(y^p;x_n^{p,n}))], \quad i,j=1,\dots,n-1, \end{aligned} \tag{3.3}$$

$$V_p=[v_i]=[x_n^{p,i}-x_n^{p,n}], \quad i=1,\dots,n-1,$$

or using (2.3) and (2.5) we can form W_p and V_p as follows

$$W_p=[w_{ij}]=[\partial_j\phi_i(y^p)-\partial_j\phi_n(y^p)], \quad i,j=1,\dots,n-1,$$

$$V_p=[v_i]=[\phi_i(y^p)-\phi_n(y^p)], \quad i=1,\dots,n-1. \tag{3.4}$$

Consider now the mapping,

$$\Lambda=(\lambda_1,\dots,\lambda_{n-1})\colon \mathscr{A}_1^*\subset\mathbb{R}^{n-1}\to\mathbb{R}^{n-1} \quad \text{by} \quad \lambda_i(y)=\phi_i(y)-\phi_n(y), \; i=1,\dots,n-1. \tag{3.5}$$

Then for the above mapping Λ and for $k=n-1$ the conditions of Theorem 3.1 are fulfilled. Consequently, the iterates y^p, $p=0,1,\dots$ of (2.12) converge to y^* and the order of convergence is two.

Suppose now that for some p, for example $p=m$, we obtain $y^m=y^*$. Then from the relationship (2.14) we can obtain that

$$x_n^{m+1}=\phi_n(y^*), \tag{3.6}$$

or

$$x_n^{m+1}=x_n^*. \tag{3.7}$$

Thus the theorem is proven. □

4. NUMERICAL APPLICATIONS

The new method described in Section 2 has been applied to random problems of varying dimensions. Our experience is that the procedure behaved predictably and reliably and the results were quite satisfactory. We present here some typical computational results obtained by Newton's method and the iterative procedure (2.12)–(2.14) applied to the following systems

$$\begin{aligned} f_1(x_1,x_2,x_3)&=x_1^3-x_1x_2x_3=0 \\ f_2(x_1,x_2,x_3)&=x_2^2-x_1x_3=0 \\ f_3(x_1,x_2,x_3)&=10x_1x_3+x_2-x_1-0.1=0 \end{aligned} \tag{4.1}$$

$$\begin{aligned} &f_1(x_1,x_2,x_3)=x_1x_3-x_3e^{x_1^2}+10^{-4}=0\\ &f_2(x_1,x_2,x_3)=x_1(x_1^2+x_2^2)+x_2^2(x_3-x_2)=0\\ &f_3(x_1,x_2,x_3)=x_1^3+x_3^3=0 \end{aligned} \tag{4.2}$$

$$\begin{aligned} &f_1(x_1,x_2,x_3,x_4,x_5)=2x_1+x_2+x_3+x_4+x_5-6=0\\ &f_2(x_1,x_2,x_3,x_4,x_5)=x_1+2x_2+x_3+x_4+x_5-6=0\\ &f_3(x_1,x_2,x_3,x_4,x_5)=x_1+x_2+2x_3+x_4+x_5-6=0\\ &f_4(x_1,x_2,x_3,x_4,x_5)=x_1+x_2+x_3+2x_4+x_5-6=0\\ &f_5(x_1,x_2,x_3,x_4,x_5)=x_1x_2x_3x_4x_5-1=0. \end{aligned} \tag{4.3}$$

System (4.1) has two roots $r_1=(0.1,0.1,0.1)$ and $r_2=(-0.1,-0.1,-0.1)$ within the cube $[-0.1,0.1]^3$ and its Jacobian at these roots is nonsingular. However, this system has a particular difficulty since the function values at some points, for example at points close to origin, cannot be accurately achieved. On the other hand, the Jacobian of system (4.2) at its root $r=(-0.99990001\cdot 10^{-4}, -0.99990001\cdot 10^{-4}, 0.99990001\cdot 10^{-4})$ is singular while the system (4.3) is a well-known test case, (*Brown's almost linear system*) [6,8]. It has roots of the form (a,a,a,a,a^{-4}), where a satisfies the equation $a^4(5a-6)+1=0$, and its Jacobian at these roots is nonsingular. The difficulty of this system is that its Jacobian at all the above roots is ill-conditioned. For this case we shall present results for the roots $r_1=(1,1,1,1,1)$, $r_2=(0.91635458253385,\ldots,1.41822708733080)$ and $r_3=(-0.57904308849412,\ldots, 8.89521544247060)$ which are reported in the following tables.

In Tables 1, 2 and 3 we exhibit the number of iterations that are required in obtaining an approximate solution of the systems (4.1), (4.2) and (4.3) correspondingly, for requiring accuracy 10^{-7} and 10^{-14} respectively, by applying Newton's method and the iterative scheme (2.12)–(2.14), for several starting points $x^0=(x_1^0,\ldots,x_n^0)$. In these tables "ε" indicates the requiring accuracy, "N" indicates the number of iterations, "FE" indicates the number of function evaluations, "AS" indicates the total number of algebraic signs that are required for applying the iterative scheme (2.17) and "r_i" denotes the root to which the corresponding method converges.

From the results shown in the tables we observe that the new method is seen to be superior to Newton's method for all the above cases (4.1)-(4.3). We observe also that the new method converges quadratically and that it converges to the same root when a different accuracy is used.

We also applied the scheme (2.12)–(2.14) to problems with precise function values for which the corresponding Jacobian was nonsingular and well-conditioned and we observed that the number of iterations of the new method was less than or equal to the corresponding number of iterations of Newton's method.

Table 1 Results for system (4.1)

Newton's method									*Scheme (2.12)–(2.14)*							
x_1^0	x_2^0	x_3^0	$\varepsilon=10^{7}$			$\varepsilon=10^{-14}$			$\varepsilon=10^{-7}$				$\varepsilon=10^{-14}$			
			N	FE	r_i	N	FE	r_i	N	FE	AS	r_i	N	FE	AS	r_i
−4	−2	1	38	456	r_1	33	396	r_2	4	36	120	r_1	5	45	150	r_1
−2	−0.5	0.2	31	372	r_1	32	384	r_1	5	45	150	r_1	6	54	180	r_1
−2	2	2	30	360	r_2	31	372	r_2	5	45	150	r_2	6	54	180	r_2
−1	−2	0.6	50	600	r_1	51	612	r_1	4	36	120	r_2	5	45	150	r_2
−1	−2	1	28	336	r_1	29	348	r_2	4	36	120	r_2	5	45	150	r_2
−0.5	0.5	−0.5	25	300	r_1	26	312	r_1	5	45	150	r_2	6	54	180	r_2
0.4	0.5	0.5	47	564	r_1	53	636	r_2	6	54	180	r_1	7	63	210	r_1
0.5	−0.5	2	27	324	r_2	28	336	r_2	4	36	120	r_2	5	45	150	r_2
0.5	2	1	33	396	r_2	54	648	r_1	5	45	150	r_2	6	54	180	r_2
2	−2	−2	27	324	r_2	43	516	r_1	4	36	120	r_2	5	45	150	r_2
5	−2	−2	29	348	r_2	38	456	r_1	6	54	180	r_2	7	63	210	r_2
10	−2	−2	38	456	r_1	39	468	r_1	7	63	210	r_2	8	72	240	r_2

Table 2 Results for system (4.2)

Newton's method							*Scheme (2.12)–(2.14)*					
x_1^0	x_2^0	x_3^0	$\varepsilon=10^{-7}$		$\varepsilon=10^{-14}$		$\varepsilon=10^{-7}$			$\varepsilon=10^{-14}$		
			N	FE	N	FE	N	FE	AS	N	FE	AS
−2	−2	−2	34	408	35	420	3	27	90	4	36	120
−1	−1	−1	30	360	31	372	2	18	60	3	27	90
−1	1	1	42	504	43	516	7	63	210	8	72	240
−0.5	−0.5	−0.5	31	372	32	384	2	18	60	3	27	90
−0.5	−0.5	0.1	23	276	26	312	2	18	60	3	27	90
0.5	0.5	0.1	44	528	45	540	2	18	60	3	27	90
0.5	0.5	0.5	28	336	30	360	2	18	60	3	27	90
1	−2	1	39	468	40	480	3	27	90	4	36	120
1	−1	1	37	444	38	456	7	63	210	8	72	240
1	1	1	46	552	47	564	6	54	180	7	63	210
2	−2	2	41	492	42	504	6	54	180	7	63	210
2	2	2	47	564	48	576	2	18	60	3	27	90

Table 3 Results for system (4.3)

Newton's method											*Scheme (2.12)–(2.14)*							
x_1^0	x_2^0	x_3^0	x_4^0	x_5^0	$\varepsilon=10^{-7}$			$\varepsilon=10^{-14}$			$\varepsilon=10^{-7}$				$\varepsilon=10^{-14}$			
					N	FE	r_i	N	FE	r_i	N	FE	AS	r_i	N	FE	AS	r_i
−8	−3	4	2	1.5	84	2520	r_3	85	2550	r_3	6	150	300	r_1	7	175	350	r_1
−4	−4	4	2	1.5	79	2370	r_3	80	2400	r_3	6	150	300	r_1	7	175	350	r_1
−2	−2	4	4	1.5	72	2160	r_3	73	2190	r_3	6	150	300	r_1	7	175	350	r_1
−1	2	−1	2	1.5	37	1110	r_3	38	1140	r_3	4	100	200	r_2	5	125	250	r_2
−0.5	−0.6	4	2	1.5	34	1020	r_3	35	1050	r_3	7	175	350	r_2	8	200	400	r_2
−0.2	−0.2	−0.2	−0.2	−0.2	35	1050	r_3	36	1080	r_3	8	200	400	r_3	9	225	450	r_3
−0.1	−0.1	−0.1	−0.1	−0.1	47	1410	r_3	48	1440	r_3	6	150	300	r_1	7	175	350	r_1
0.1	−0.1	0.1	−0.1	4	43	1290	r_3	41	1230	r_3	6	150	300	r_1	7	175	350	r_1
0.1	0.1	0.1	0.1	0.1	47	1410	r_3	48	1440	r_3	6	150	300	r_1	7	175	350	r_1
0.1	0.1	0.1	0.1	0.2	45	1350	r_1	46	1380	r_1	6	150	300	r_1	7	175	350	r_1
0.1	0.1	0.1	0.1	1	34	1020	r_1	35	1050	r_1	6	150	300	r_1	7	175	350	r_1
3	3	3	4	1.5	80	2400	r_3	72	2160	r_3	6	150	300	r_1	7	175	350	r_1
10	−3	1.5	−3	1.5	80	2400	r_3	81	2430	r_3	6	150	300	r_1	7	175	350	r_1
10	3	4	2	1.5	82	2460	r_3	83	2490	r_3	6	150	300	r_1	7	175	350	r_1

5. CONCLUDING REMARKS

The method we have analysed in this paper compares favourably with Newton's method when the Jacobian of F at the root of the system (1.1) is singular or ill-conditioned or when the values of the components of F cannot be accurately achieved.

Also although our method uses reduction to simpler one-dimensional equations, it converges quadratically to $n-1$ components of the solution, while the remaining component of the solution is evaluated separately using the final approximations of the other components. Thus it does not require a good initial estimate for one component of the solution.

Moreover, the method does not directly perform function evaluations, and also using the iterative scheme (2.17) it requires only their algebraic signs to be correct in finding the various $\phi_i(y)$.

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