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A SHORT PROOF AND A GENERALIZATION OF MIRANDA'S EXISTENCE THEOREM

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ABSTRACT. Miranda gave in [5] an equivalent formulation of the famous Brouwer fixed point theorem. We give a short proof of Miranda's existence theorem and then using the results obtained in this proof we give a generalization of a well-known variant of Bolzano's existence theorem. Finally, we give a generalization of Miranda's theorem.

We shall give here a short proof and a generalization of the following equivalent formulation of the famous L. E. J. Brouwer fixed point theorem [1] given by C. Miranda [5].

Theorem 1 (Miranda, 1940) [5, 10, 4, 6, 3, 11]. *Let* $G = \{x \in \mathbf{R}^n : |x_i| < L,$ *for* $1 \leq i \leq n\}$ *and suppose that the mapping* $F = (f_1, f_2, \ldots, f_n): \overline{G} \to \mathbf{R}^n$ *is continuous on the closure* $\overline{G}$ *of* G *such that* $F(x) \neq \theta = (0, 0, \ldots, 0)$ *for* x *on the boundary* ϑG *of* G, *and*

(i) $f_i(x_1, x_2, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \geq 0$ *for* $1 \leq i \leq n$, *and*
(ii) $f_i(x_1, x_2, \ldots, x_{i-1}, +L, x_{i+1}, \ldots, x_n) \leq 0$ *for* $1 \leq i \leq n$.

Then, $F(x) = \theta$ *has a solution in* G.

For recent proofs of the above theorem see [10, pp. 37–38] and [3, pp. 118–119]. Theorem 1 is known to be useful in the theory of differential equations. Moreover, for some of its implementations in the case of systems of nonlinear algebraic or transcendental equations, we refer to [4, 6, 11]. Theorem 1, also , has an important property since it constitutes a straightforward generalization of the well-known and very useful, (for iterative approximate procedures for solving nonlinear equations), Bolzano's existence theorem which states: "*If* $f: [a, b] \to \mathbf{R}$ *is a continuous mapping and* $f(a)$ *and* $f(b)$ *have opposite signs, then for some* $x \in (a, b)$, *it holds* $f(x) = 0$".

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We now give a short proof of Theorem 1. For the details about degree theory used in the following proof, we refer to [7, 2, 9, 8, 3] .

Proof of Theorem 1. Consider the homotopy,

$$H: \overline{G} \times [0,1] \subset \mathbf{R}^{n+1} \to \mathbf{R}^n, \qquad \text{by } H(x,t) = (1-t)F(x) + t(-x).$$

Then, $H(x,t) \neq \theta$ for $(x,t) \in \vartheta G$ and $t \in [0,1]$. In fact, $H(x,0) = F(x) \neq \theta$ since $\theta \notin F(\vartheta G)$, while $H(x,1) = -x \neq \theta$ since $\theta \notin \vartheta G$; finally, $H(x,t) = \theta$ for some $t \in (0,1)$ leads to the contradiction $F(x) + t(1-t)^{-1}(-x) = \theta$ because $t(1-t)^{-1} > 0$ and by the assumptions (i) and (ii) for $x \in \vartheta G$ there exist at least one i such that $f_i(x)(-x_i) > 0$. Thus by the homotopy invariance theorem of the degree theory, it follows that

$$\deg[F, G, \theta] = \deg[H(\cdot, 0), G, \theta] = \deg[H(\cdot, 1), G, \theta],$$

(where $\deg[F, G, \theta]$ indicates the topological degree of F at θ relative to G). Hence, $|\deg[F, G, \theta]| = 1 \neq 0$ and the result follows by the Kronecker existence theorem. □

A corollary directly derived from the above result follows:

Corollary 1. *Suppose that the conditions of the preceding theorem hold. Assume that $F(x) = \theta$ has only simple solutions in G, (i.e., the Jacobian determinant of F does not vanish at any solution). Then $F(x) = \theta$ has an odd number of solutions in G.*

Proof. The result follows from the fact that $|\deg[F, G, \theta]| = 1$, which we have determined in the proof of the previous theorem. □

It is readily seen that Corollary 1 generalizes a well-known variant of Bolzano's Theorem (odd number of solutions) which states : "*If $f(a)$ and $f(b)$ have opposite signs and whenever $f(x) = 0$ for $x \in (a,b)$ holds that $f'(x) \neq 0$, then $f(x) = 0$ has an odd number of solutions in (a,b)*".

A generalization of Theorem 1 follows:

Theorem 2. *Let $\beta_1, \beta_2, \ldots, \beta_n$ be n linearly independent vectors in $\mathbf{R}^n$, let $\langle \cdot, \cdot \rangle$ denote the standard inner product and $G = \{x \in \mathbf{R}^n : |\langle \beta_i, x \rangle| < L$, for $1 \leq i \leq n\}$. Suppose that $F = (f_1, f_2, \ldots, f_n): \overline{G} \to \mathbf{R}^n$ is a continuous mapping such that $F(x) \neq \theta$ for $x \in \vartheta G$, and*

$$\langle F(x), \beta_i \rangle \geq 0 \quad \text{if} \quad \beta_i^\top x = -L \quad \text{for} \quad 1 \leq i \leq n, \quad \text{and}$$

$$\langle F(x), \beta_i \rangle \leq 0 \quad \text{if} \quad \beta_i^\top x = +L \quad \text{for} \quad 1 \leq i \leq n.$$

Then, $F(x) = \theta$ has a solution in G and, in fact, $|\deg[F, G, \theta]| = 1$.

Proof. Consider the mapping,

$$\Lambda : \mathbf{R}^n \to \mathbf{R}^n, \quad \text{by} \quad \Lambda(x) = (\langle \beta_1, x \rangle, \langle \beta_2, x \rangle, \ldots, \langle \beta_n, x \rangle).$$

Clearly, Λ is a one-to-one linear mapping. So,

$$\deg[F, G, \theta] = \deg[\Lambda F \Lambda^{-1}, \Lambda G, \theta],$$

which reduces the present theorem to Theorem 1. Finally, following the proof of Theorem 1 we can obtain $|\deg[F, G, \theta]| = 1$. □

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