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VRAHATIS, M. N.; GRAPSA, T. N.; RAGOS, O.; ZAFIROPOULOS, F. A.

On the Localization and Computation of Zeros of Bessel Functions

The topological degree of a continuous mapping is implemented for the calculation of the total number of the simple real zeros within any interval of the Bessel functions of first and second kind and their derivatives. A new algorithm, based on this implementation, is given for the localization and isolation of these zeros. Furthermore, a second algorithm is presented for their computation employing a modified bisection method. The only information required for this computation is the algebraic signs of function values. Moreover, lower and upper bounds of a zero can also be obtained.

MSC (1991): 33C10, 65D20, 34B30

1. Introduction

The study of the Helmholtz equation in cylindrical coordinates eventually leads to the well-known *Bessel equation*

$$x^2u''(x) + xu'(x) + (x^2 - \nu^2)\,u(x) = 0\,, \tag{1}$$

the solutions of which are

a) the *Bessel function of order ν and argument x of the first kind* given by the series

$$J_\nu(x) = \frac{x^\nu}{2^\nu} \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{2^{2m} m!\, \Gamma(\nu + m + 1)}\,, \tag{2}$$

which converges for every x and

b) the *Bessel function of order ν and argument x of the second kind* expressible in terms of functions of the first kind by the formula

$$Y_\nu(x) = \begin{cases} \dfrac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}\,, & \nu \text{ non-integral}\,, \\ \lim\limits_{n\to\nu} \dfrac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}\,, & \nu = 0,\, 1,\, 2,\, \ldots\,. \end{cases} \tag{3}$$

Regarding the zeros j_ν, j'_ν, y_ν, and y'_ν of the functions $J_\nu(x)$, $J'_\nu(x)$, $Y_\nu(x)$, and $Y'_\nu(x)$, respectively, the following holds (cf., e.g., [5, 28]): *For any real value of ν these functions have infinitely many real zeros all of which are simple, with the possible exception of $x = 0$. Especially, for $\nu \geq -1$, $J_\nu(x)$ has only real zeros. If $\nu \geq 0$, all zeros of $J'_\nu(x)$ are real.*

For non-negative ν, the k-th positive zeros of the above functions are denoted correspondingly by $j_{\nu,k}$, $j'_{\nu,k}$, $y_{\nu,k}$, and $y'_{\nu,k}$. They are interlaced according to the inequalities

$$j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \cdots\,,$$

$$y_{\nu,1} < y_{\nu+1,1} < y_{\nu,2} < y_{\nu+1,2} < y_{\nu,3} < \cdots\,,$$

$$\nu \leq j'_{\nu,1} < y_{\nu,1} < y'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < y_{\nu,2} < y'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < \ldots\,.$$

The mathematical problem of localizing and computing zeros of Bessel functions, encountered in many fields in science and engineering, has drawn a lot of attention and has evolved to a rather specialized branch of mathematics.

In the present paper we implement the concept of the topological degree to calculate the total number of real roots of Bessel functions within a predetermined interval and to isolate each one of them. For this purpose we use PICARD's extension [16, 17, 7, 8, 21] and either Kronecker theory or KEARFOTT's degree computation method. Once a zero is isolated, it can be computed numerically, utilizing a modified bisection method, to any accuracy (subject to relative machine precision). Thus, we propose two algorithms, one for the isolation and one for the computation of a real zero of a Bessel function. Upper and lower bounds for any root can also be produced.

2. The topological degree for the localization of zeros

Definition 1: Suppose that the function $F_n = (f_1, \ldots, f_n) : \bar{\mathscr{D}}^n \subset \mathbb{R}^n \to \mathbb{R}^n$ is twice continuously differentiable in the domain $\mathscr{D}^n$ the boundary of which is denoted by $b(\mathscr{D}^n)$. Suppose further that the solutions of the equation

$$F_n(x) = \mathcal{O}_n\,, \tag{4}$$

where $\mathcal{O}_n = (0, \ldots, 0)$ denotes the origin of $\mathbb{R}^n$, are not located on $b(\mathscr{D}^n)$, and they are simple, i.e., the Jacobian determinant of F_n at these solutions is non-zero. Then the *topological degree of* F_n *at* $\mathcal{O}_n$ *relative to* $\mathscr{D}^n$ is denoted by $\deg[F_n, \mathscr{D}^n, \mathcal{O}_n]$ and can be defined by the following sum:

$$\deg[F_n, \mathscr{D}^n, \mathcal{O}_n] = \sum_{x \in F_n^{-1}(\mathcal{O}_n)} \operatorname{sgn} J_{F_n}(x), \tag{5}$$

where J_{F_n} indicates the determinant of the Jacobian matrix, and sgn defines the well known sign function with values

$$\operatorname{sgn} \psi = \begin{cases} -1, & \text{if} \quad \psi < 0, \\ 0, & \text{if} \quad \psi = 0, \\ 1, & \text{if} \quad \psi > 0. \end{cases}$$

The above definition can be generalized when F_n is only continuous [6, 14].

Kronecker's theorem [2, 6, 14] states that equation (4) has at least one root in $\mathscr{D}^n$ if $\deg[F_n, \mathscr{D}^n, \mathcal{O}_n] \neq 0$.

The definition of the topological degree actually indicates that its value is equal to the number of simple solutions of equation (4) for which the Jacobian determinant is positive, minus the number of simple solutions for which the Jacobian is negative. Evidently, if all of them give the same Jacobian sign, then the total number $\mathscr{N}^r$ of simple roots of $F_n(x)$ can be obtained by the value of $\deg[F_n, \mathscr{D}^n, \mathcal{O}_n]$. To this end PICARD has considered the following extensions of the function F_n and the domain $\mathscr{D}^n$:

$$F_{n+1} = (f_1, \ldots, f_n, f_{n+1}) : \mathscr{D}^{n+1} \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \tag{6}$$

where $f_{n+1} = yJ_{F_n}$ and $\mathscr{D}^{n+1}$ is the direct product of the domain $\mathscr{D}^n$ with an arbitrary interval of the real y-axis containing the point $y = 0$. Then the following system of equations,

$$\begin{aligned} &f_i(x_1, x_2, \ldots, x_n) = 0, \qquad i = 1, \ldots, n, \\ &yJ_{F_n}(x_1, x_2, \ldots, x_n) = 0, \end{aligned} \tag{7}$$

possesses the same simple roots with $F_n(x)$, provided $y = 0$. Also, it is easily seen that the Jacobian of (7) is equal to $J^2_{F_n(x)}$ which is always positive. Thus we conclude that the total number $\mathscr{N}^r$ of solutions of equation (4) is

$$\mathscr{N}^r = \deg[F_{n+1}, \mathscr{D}^{n+1}, \mathcal{O}_{n+1}]. \tag{8}$$

2.1 Kronecker integral approach

The topological degree can be represented by the Kronecker integral as follows:

$$\deg[F_n, \mathscr{D}^n, \mathcal{O}_n] = \frac{1}{\Omega_n} \int \int_{b(\mathscr{D}^n)} \cdots \int \frac{\sum_{i=1}^{n} A_i \, \mathrm{d}x_1 \ldots \mathrm{d}x_{i-1} \, \mathrm{d}x_{i+1} \ldots \mathrm{d}x_n}{(f_1^2 + f_2^2 + \ldots + f_n^2)^{n/2}}, \tag{9}$$

where A_i define the following determinants:

$$A_i = (-1)^{n(i-1)} \begin{vmatrix} f_1 & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{i-1}} & \frac{\partial f_1}{\partial x_{i+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ f_2 & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_{i-1}} & \frac{\partial f_2}{\partial x_{i+1}} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_n & \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_{i-1}} & \frac{\partial f_n}{\partial x_{i+1}} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix},$$

and Ω_n denotes the surface of a hypersphere in $\mathbb{R}^n$ with radius one, i.e.: $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

The Kronecker integral has been numerically approximated with Gauss-Legendre quadrature [13].

In the present paper we study the zeros of a Bessel function, so we focus on the problem of calculating the total number of simple roots of a real function $f(x)$, defined in a predetermined interval $[a, b]$ and twice continuously differentiable in (a, b), where a and b are arbitrarily chosen so that $f(a) f(b) \neq 0$.

According to Picard's extension we consider the function $F_2 = (f_1, f_2) : \mathscr{P}^2 \subset \mathbb{R}^2 \to \mathbb{R}^2$ and the corresponding system

$$f_1(x, y) = f(x) = 0, \qquad f_2(x, y) = yf'(x) = 0, \tag{10}$$

where the prime denotes differentiation, and $\mathscr{P}^2$ is the rectangular parallelepiped $[a, b] \times [-\xi, \xi]$ in the (x, y)-plane with ξ an arbitrary positive constant. Since the roots are simple, which means $f'(x) \neq 0$ for $x \in f^{-1}(0)$, it is easily seen that the solutions of system (10) in $\mathscr{P}^2$, and those of $f(x) = 0$ in (a, b) are the same. Also, since $J_{F_2} = f'^2$, the total

number of simple zeros $\mathcal{N}^r$ of $f(x)$ in (a, b) is given by

$$\mathcal{N}^r = \deg[F_2, \mathcal{P}^2, \mathcal{O}_2]. \tag{11}$$

For the computation of the topological degree of F_2 we apply Kronecker's integral (9) for $n = 2$. Using the relations $\mathrm{d}f_j = \frac{\partial f_j}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial f_j}{\partial x_2}\,\mathrm{d}x_2$, $j = 1, 2$, we obtain

$$\mathcal{N}^r = \frac{1}{2\pi} \oint_{b(\mathcal{P}^2)} \frac{f_1\,\mathrm{d}f_2 - f_2\,\mathrm{d}f_1}{f_1^2 + f_2^2} = \frac{1}{2\pi} \oint_{b(\mathcal{P}^2)} \mathrm{d}\arctan\left(\frac{f_2}{f_1}\right). \tag{12}$$

Replacing f_1 and f_2 by virtue of (10) and performing the integration in (12) we finally get

$$\mathcal{N}^r = -\frac{1}{\pi}\left[\xi \int_a^b \frac{f(x)\,f''(x) - f'^2(x)}{f^2(x) + \xi^2 f'^2(x)}\,\mathrm{d}x - \arctan\left(\frac{\xi f'(b)}{f(b)}\right) + \arctan\left(\frac{\xi f'(a)}{f(a)}\right)\right]. \tag{13}$$

Remark 1: It has been explicitly shown by PICARD [16, 17] that relation (13) is independent of the value of ξ.

The above developed method is applied in the sequel for the localization of the simple real zeros of the Bessel functions of the first kind, $J_\nu(x)$. In this case relation (13) becomes

$$\mathcal{N}^r = -\frac{1}{\pi}\left[\xi \int_a^b G(x)\,\mathrm{d}x - \arctan\left(\frac{\xi J'\nu(b)}{J_\nu(b)}\right) + \arctan\left(\frac{\xi J'_\nu(a)}{J_\nu(a)}\right)\right], \tag{14}$$

where the function $G(x)$, by means of equation (1), can be expressed in the following form:

$$G(x) = \frac{\left(\frac{\nu^2}{x^2} - 1\right) J_\nu^2(x) - \frac{1}{x}\,J_\nu(x)\,J'_\nu(x) - J'^2_\nu(x)}{J_\nu^2(x) + \xi^2 J'^2_\nu(x)}. \tag{15}$$

2.2 KEARFOTT's approach

We could use any one of the degree computation methods (see e.g. [22, 13, 10, 11, 23, 24]) to determine the total number of zeros by virtue of equation (11). Here we use KEARFOTT's method [10, 11, 12] which compares favorably to other methods, in efficiency. This method is briefly described below.

Suppose that $S^{n-1} = \langle x_1, x_2, \ldots, x_n\rangle$ is an $(n-1)$-simplex [22, 10, 11] in $\mathbb{R}^n$ and assume $F_n = (f_1, f_2, \ldots, f_n) : S^{n-1} \to \mathbb{R}^n$ is continuous. Then the *range simplex associated with* S^{n-1} *and* F_n, denoted by $\mathcal{R}(S^{n-1}, F_n)$, is an $n \times n$ matrix with elements ϱ_{ij}, $1 \le i, j \le n$, given by

$$\varrho_{ij} = \begin{cases} 1, & \text{if } f_j(x_i) \ge 0, \\ -1, & \text{if } f_j(x_i) < 0. \end{cases} \tag{16}$$

$\mathcal{R}(S^{n-1}, F_n)$ is called *usable* if one of the following conditions hold:

a) the elements ϱ_{ij} of $\mathcal{R}(S^{n-1}, F_n)$, are:

$$\varrho_{ij} = \begin{cases} 1, & \text{if } i \ge j, \\ -1, & \text{if } j = i + 1. \end{cases} \tag{17}$$

b) $\mathcal{R}(S^{n-1}, F_n)$ can be put into this form by a permutation of its rows.

When $\mathcal{R}(S^{n-1}, F_n)$ is usable, then the *parity* $\mathrm{Par}(\mathcal{R}(S^{n-1}, F_n))$ is defined to be 1, if the number of the permutations of the rows required to put $\mathcal{R}(S^{n-1}, F_n)$ into the form (17) is even. If this number is odd then $\mathrm{Par}(\mathcal{R}(S^{n-1}, F_n))$ is defined to be -1. For all other cases, we set $\mathrm{Par}(\mathcal{R}(S^{n-1}, F_n)) = 0$. Suppose that $\mathcal{P}^n$ is an n-dimensional polyhedron for some $n \ge 2$, and that $\{S_i^{n-1}\}_{i=1}^m$ is a finite set of $(n-1)$-simplexes with disjoint interiors such that $\sum_{i=1}^m S_i^{n-1} = b(\mathcal{P}^n)$; then, under some assumptions regarding S_i^{n-1}, the value of the topological degree of F_n at $\mathcal{O}_n$ relative to $\mathcal{P}^n$ can be obtained by the following relation:

$$\deg[F_n, \mathcal{P}^n, \mathcal{O}_n] = \sum_{i=1}^m \mathrm{Par}(\mathcal{R}(S_i^{n-1}, F_n)). \tag{18}$$

Remark 2: Kearfott's degree computation method is very efficient and has the advantage that it requires only the signs of function values to be correct.

2.3 Isolating a zero

With the above discussion in mind we give a description of our algorithm, in "pseudo-Pascal" (see e.g. [15]), independently of the degree computation method. In this algorithm $F_\nu(x)$ indicates the considered Bessel function of order ν, and (a, b) is a predetermined interval. It results in the total number $\mathcal{N}^r$ of roots which exist within (a, b). Besides, it isolates one of them, bisecting the initial interval, and gives a lower bound a_k and an upper one, b_k, for this root. If the isolation of the rest of them is required, the whole procedure has to be repeated successively for the remaining intervals (a, a_k) and (b_k, b).

```
Algorithm degree_isolate (a, b, N^r);
{comment: This algorithm results in the total number N^r of roots of F_ν(x) in (a, b) and isolates one of them}.
procedure degree (a_k, b_k, N^r_k); {comment: computes the total number of zeros}
  begin
    Find N^r_k, the number of zeros in (a_k, b_k);
  end {degree}
procedure isolation (a_k, b_k, N^r_k); {comment: isolates a zero of F_ν(x) in (a_k, b_k)}
  begin
    while N^r_k > 1 do
      begin
        degree (a_k, (a_k + b_k)/2, N^r_1);
        N^r_2 := N^r_k − N^r_1;
        if N^r_2 = 0 then isolation (a_k, (a_k + b_k)/2, N^r_1);
        if N^r_1 < N^r_2 and N^r_1 ≠ 0 then isolation (a_k, (a_k + b_k)/2, N^r_1)
        else isolation ((a_k + b_k)/2, b_k, N^r_2);
      end {while}
  end {isolation}
begin {degree_isolate}
  input a, b
  degree (a, b, N^r);
  N^r_* := N^r;
  if N^r > 1 then isolation (a, b, N^r);
  output a, b, N^r_*
end. {degree_isolate}
```

3. Computing roots of Bessel functions

Having isolated one root of a Bessel function within an interval, we can use a modified version of the bisection method to compute it, as described in [25, 26]. It is reported there that, in order to compute a solution of $f(x) = 0$, where $f: [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuous, the following iterative formula can be used:

$$x_{i+1} = x_i + c \operatorname{sgn} f(x_i)/2^{i+1}, \qquad i = 0, 1, \ldots, \tag{19}$$

with $x_0 = a$ and $c = \operatorname{sgn} f(a)\,(b - a)$. The iterations (19) converge to a root $r \in (a, b)$ if for some x_i, $i = 1, 2, \ldots$, there holds

$$\operatorname{sgn} f(x_0) \operatorname{sgn} f(x_i) = -1.$$

The number of iterations, η, which are required in obtaining an approximate root r^* such that $|r - r^*| \le \varepsilon$ for some $\varepsilon \in (0, 1)$, is given by

$$\eta = \lceil \log_2 ((b - a)\,\varepsilon^{-1}) \rceil, \tag{20}$$

where the notation $\lceil \cdot \rceil$ refers to the smallest integer not less than the real number quoted.

It is evident from (19) that the only computable information required by the bisection method consists in the algebraic signs of the function f; so it can be applied to problems with imprecise function values. Moreover, the bisection method is a globally convergent method, it always converges within the given interval, and it is optimal [19, 20] in the sense that it possesses asymptotically the best rate of convergence. Also, it can be efficiently implemented for the computation of all the zeros and extrema of a function [9].

To apply our method, as previously mentioned, we only need the algebraic signs of the function values to be correct. The following results suggest the number of terms necessary in order to determine the algebraic signs of $J_\nu(x)$.

Proposition 1: *The sign of the Bessel function J_ν for $\nu > -1$ is the same as the sign of the sum*

$$L_\nu(x) = \frac{x^\nu}{2^\nu} \sum_{m=0}^{M+K+1} (-1)^m a_m, \tag{21}$$

where a_m *are given by*

$$a_m = \frac{x^{2m}}{2^{2m} m!\, \Gamma(\nu + m + 1)}, \tag{22}$$

with

$$M = \left\lceil \frac{-\nu - 2 + \sqrt{\nu^2 + x^2}}{2} \right\rceil, \tag{23}$$

and K *is defined so that the following relation holds:*

$$|a_{M+K+1}| < \left| \sum_{k=0}^{M+K} (-1)^k a_k \right|. \tag{24}$$

Proof: We observe that

a) $a_m > 0$,

b) $\lim_{m \to \infty} a_m = 0$,

c) $a_m \geq a_{m+1} \Leftrightarrow 4(m+1)(\nu + m + 1) > x^2$.

Thus, the series

$$R_\nu(x) = \sum_{m=M}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!\, \Gamma(\nu + m + 1)} \tag{25}$$

is an alternating series which satisfies the three conditions of Leibniz's Theorem provided that

$$m > \frac{-\nu - 2 + \sqrt{\nu^2 + x^2}}{2}. \tag{26}$$

The Alternating Series Estimation Theorem ensures that the absolute value of the $(M + K + 1)$-th term is larger than that of the remainder of the series $R_\nu(x)$. Furthermore, the signs of the $(M + K + 1)$-th term and of the remainder of the series $R_\nu(x)$ are the same. Now, by assumption (24) (which is fulfilled for some term a_i since the sequence $\{a_i\}$ tends to zero when $i \to \infty$), the sign of the Bessel function $J_\nu(x)$ is the same as the corresponding sign of $L_\nu(x)$. Thus the proposition is proved.

We can now give a description of our algorithm that computes, within a predetermined accuracy ε, a real root of the Bessel function $J_\nu(x)$, which has been isolated in the interval (a_k, b_k).

```
Algorithm compute_zero (a, b);
{comment: This algorithm computes a zero of J_ν(x) in (a, b). It uses (19) and requires Γ(x), ε}.
procedure sign (x, s); {comment: finds the sign s of J_ν(x)}
  begin
    M := ⌈(−ν − 2 + √(ν² + x²))/2⌉;
    S := 0;
    for m := 0 to M do S := S + (−1)^m x^{2m+ν}/(2^{2m+ν} m! Γ(ν + m + 1));
    m := M + 1;
    T := (−1)^m x^{2m+ν}/(2^{2m+ν} m! Γ(ν + m + 1));
    while |T| ≥ |S| do
      begin
        S := S + T;
        m := m + 1;
        T := (−1)^m x^{2m+ν}/(2^{2m+ν} m! Γ(ν + m + 1));
      end {while}
    Find s := sign (S + T);
  end {sign}
begin {compute_zero}
  input a_k, b_k, ε
  x_0 := a_k;
  sign (x_0, s_0);
  c := s_0(b_k − a_k);
  for i := 0 to ⌈log_2 ((b_k − a_k) ε^{−1})⌉ do
    begin
      sign (x_i, s_i);
      x_{i+1} := x_i + s_i c/2^{i+1};
    end
  output x_{i+1}
end. {compute_zero}
```

Remark 3: In order to obtain zeros of other Bessel functions, the above algorithm has to be adapted accordingly.

Evidently, utilizing the above results we are able to produce bounds for any real zero of a Bessel function. The following proposition proves it.

Proposition 2: *Suppose that* $f: [a, b] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function and* (a, b) *contains only one simple zero* x^* *of* f. *Consider the sequence* (19) *and its subsequences* $\{x_i^-\}$, $\{x_i^+\}$ *such that* $\operatorname{sgn} f(x_i^-) = \operatorname{sgn} f(a)$, $\operatorname{sgn} f(x_i^+) = -\operatorname{sgn} f(a) = \operatorname{sgn} f(b)$. *Then,* a_η *and* b_η, *determined so that*

$$a_\eta = \max_{0 \le i \le \eta} \{x_i^-\}, \qquad b_\eta = \min_{0 \le i \le \eta} \{x_i^+\}, \tag{27}$$

where η *is given by equation* (20), *for any* $\varepsilon \in (0, 1)$, *constitute a lower and an upper bound of the zero* x^* *with* $b_\eta - a_\eta < \varepsilon$.

Remark 4: The function f can be replaced by any of the Bessel functions considered in this paper.

4. Applications

We have tested our algorithms with several random intervals (a, b) and various Bessel functions. In each case we have calculated the total number $\mathcal{N}^r$ of roots of the respective Bessel function existing within (a, b). For this calculation we have used both Kronecker's integral applied to Picard's extension and Kearfott's method. The computation of the integral of equation (14) depends on the integration method used. We have tried various integration methods. Here we exhibit the results obtained by Romberg's integration method. Also, we have observed that the speed of the numerical computation of the integral in equation (14) depends on the value of ξ. Our experience is that the fastest computation (using Romberg's method) can be obtained for $0.8 \le \xi \le 1.2$ while for other values the computational time increases (see Fig. 1). Kearfott's method is independent of the choice of ξ, but its speed depends on a stopping parameter p and the maximum tree depth MD [11]. In our case, we have obtained accurate results even for small values of p and MD, as for instance $p = 1$, MD $= 2$.

Table 1 presents the number of zeros of Bessel functions of various orders ν existing within some given intervals (a, b), as well as the respective subintervals (a_k, b_k), where exactly one root, r_k, exists.

In Table 2 we give the first ten zeros of several Bessel functions, chosen at random, computed by Algorithm *compute_zero*.

Table 1

	a	b	$\mathcal{N}^r$	a_k	b_k	r_k
J_0	1	100	32	1.0000	4.09375	2.404 825 557 695 773
$J_{0.3}$	1	100	31	81.4375	87.625 00	84.509 788 949 453 324
$J_{2.5}$	1	100	30	1.0000	7.187 50	5.763 459 196 894 550
J_{10}	10	100	27	10.0000	15.625 00	14.475 500 686 554 542
$J_{13.3}$	10	100	25	10.0000	21.250 00	18.131 465 204 981 082
J_{50}	50	100	11	50.0000	62.500 00	57.116 899 160 119 190
J_0'	1	100	31	13.3750	19.562 50	16.500 922 441 528 084
$J_{0.3}'$	1	100	32	1.0000	4.093 75	1.308 699 363 719 847
$J_{2.5}'$	1	100	31	1.0000	7.187 50	3.959 527 916 501 095
J_{10}'	10	100	27	10.0000	15.625 00	12.128 927 704 415 439
$J_{13.3}'$	10	100	26	10.0000	15.625 00	15.617 873 137 336 602
J_{50}'	50	100	11	50.0000	56.250 00	53.502 858 820 400 364
Y_0	1	100	31	13.3750	19.562 50	16.470 630 050 877 633
$Y_{0.3}$	1	100	31	1.0000	7.187 50	4.301 991 992 307 308
$Y_{2.5}$	1	100	31	1.0000	7.187 50	3.632 797 319 831 763
Y_{10}	10	100	27	10.0000	15.625 00	11.770 876 674 955 581
$Y_{13.3}$	10	100	26	10.0000	15.625 00	15.242 738 650 014 224
Y_{50}	50	100	11	50.0000	56.250 00	52.997 640 387 316 651
Y_0'	1	100	32	1.0000	4.093 75	2.197 141 326 031 016
$Y_{0.3}'$	1	100	31	81.4375	87.625 00	84.503 872 394 936 382
$Y_{2.5}'$	1	100	30	1.0000	7.187 50	5.634 296 563 929 559
Y_{10}'	10	100	27	10.0000	15.625 00	14.353 013 743 699 869
$Y_{13.3}'$	10	100	25	10.0000	21.250 00	18.006 479 815 952 494
Y_{50}'	50	100	11	50.0000	62.500 00	56.962 904 275 167 517

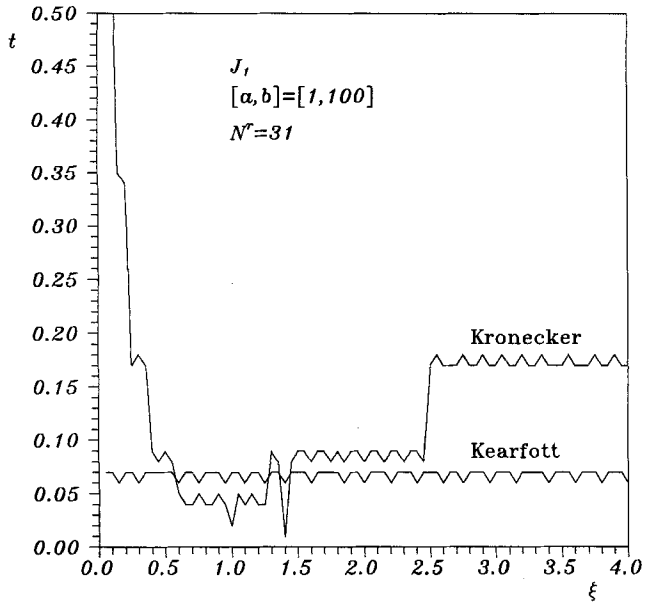

Fig. 1. Time required for the computation of the topological degree using Kronecker's integral versus ξ and comparison with Kearfott's method

Table 2. Zeros of various Bessel functions computed by Algorithm *compute_zero*

J_0	2.404 825 557 695 773	$J_{2.5}$	5.763 459 196 894 550	J_{50}	57.116 899 160 119 190
	5.520 078 110 286 310		9.095 011 330 476 354		62.807 698 764 835 380
	8.653 727 912 911 011		12.322 940 970 566 584		67.697 408 410 764 783
	11.791 534 439 014 282		15.514 603 010 886 749		72.190 366 544 011 145
	14.930 917 708 487 784		18.689 036 355 362 817		76.437 072 182 667 940
	18.071 063 967 910 918		21.853 874 222 709 777		80.513 239 317 465 752
	21.211 636 629 879 254		25.012 803 202 289 602		84.463 252 949 061 289
	24.352 471 530 749 303		28.167 829 707 993 626		88.315 711 749 199 524
	27.493 479 132 040 244		31.320 141 707 447 189		92.090 274 456 391 647
	30.634 606 468 431 967		34.470 488 331 285 007		95.801 108 265 953 272
J_0'	3.957 678 419 314 857	$J_{2.5}'$	3.959 527 916 501 094	J_{50}'	53.502 858 820 400 364
	7.086 051 060 301 773		7.451 610 064 214 504		60.112 444 427 740 558
	10.222 345 043 496 417		10.715 647 375 791 513		65.317 141 149 297 498
	13.361 097 473 872 762		13.921 686 012 308 782		69.981 432 989 702 090
	16.500 922 441 528 084		17.103 359 117 208 743		74.338 747 166 755 810
	19.641 309 700 887 942		20.272 369 140 216 535		78.493 210 918 315 127
	22.782 028 047 291 558		23.433 926 142 067 801		82.501 961 830 679 640
	25.922 957 653 180 932		26.590 716 631 086 269		86.400 289 204 160 094
	29.064 030 252 728 390		29.744 270 680 556 565		90.211 743 342 930 555
	32.205 204 116 493 269		32.895 525 188 224 304		93.952 927 913 983 330
Y_0	3.831 705 970 207 513	$Y_{2.5}$	3.632 797 319 831 762	Y_{50}	52.997 640 387 316 651
	7.015 586 669 815 619		7.367 008 971 566 918		60.026 319 332 799 442
	10.173 468 135 062 721		10.663 561 390 481 999		65.272 723 327 026 940
	13.323 691 936 314 222		13.883 369 775 209 742		69.951 692 734 372 273
	16.470 630 050 877 633		17.072 848 832 681 667		74.316 349 977 805 003
	19.615 858 510 468 247		20.246 944 819 393 926		78.475 187 454 486 474
	22.760 084 380 592 769		23.412 099 866 421 985		82.486 831 848 982 885
	25.903 672 087 618 386		26.571 579 024 683 013		86.387 212 778 756 663
	29.046 828 534 916 836		29.727 222 608 526 206		90.200 199 617 262 318
	32.189 679 910 974 398		32.880 149 822 147 309		93.942 572 252 611 562
Y_0'	2.197 141 326 031 016	$Y_{2.5}'$	5.634 296 563 929 560	Y_{50}'	56.962 904 275 167 517
	5.429 681 040 794 136		9.030 901 729 624 809		62.748 881 669 459 305
	8.596 005 868 331 170		12.278 862 551 656 813		67.661 781 418 302 983
	11.749 154 830 839 879		15.480 654 965 172 806		72.164 827 720 771 193
	14.897 442 128 336 724		18.661 308 999 166 464		76.417 109 157 783 180
	18.043 402 276 727 858		21.830 389 858 741 036		80.496 796 665 401 667
	21.188 068 934 142 211		24.992 411 488 536 626		84.449 230 155 149 340
	24.331 942 571 356 901		28.149 798 292 577 922		88.303 453 538 378 357
	27.475 294 980 449 235		31.303 973 776 577 955		92.079 360 103 387 758
	30.618 286 491 641 110		34.455 830 546 321 302		95.791 251 970 104 038

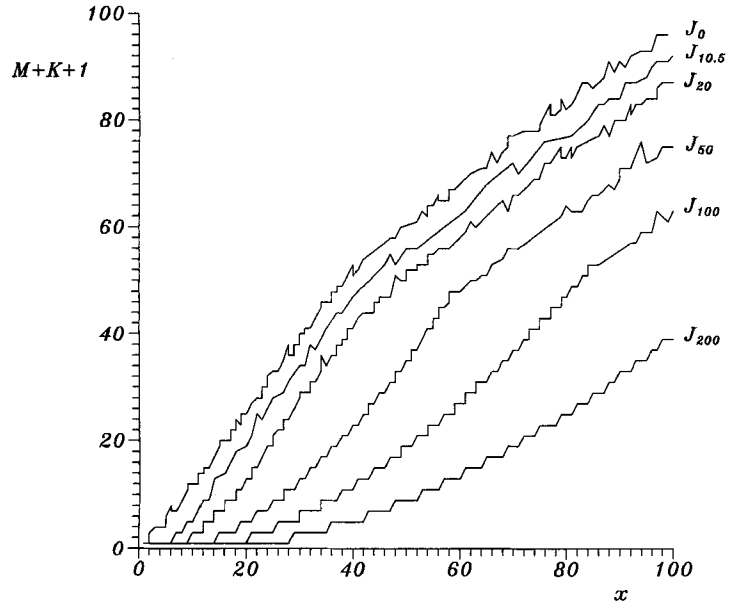

Fig. 2. The number of terms to be added for the determination of the sign of $J_\nu(x)$ versus x for $\nu = 0, 10.5, 20, 50, 100, 200$

By virtue of Proposition 1, we have obtained the number of terms of J_ν required to obtain the corresponding algebraic sign for various values of the order ν and the argument x. As it is shown in Fig. 2, the number of terms decreases as the order increases.

5. Concluding remarks

An efficient method for locating, isolating, and computing real zeros of Bessel functions is described in this paper. It has been implemented and tested, and our experience is that it behaves predictably and accurately.

The first phase of this method (algorithm *degree_isolate*) exploits topological degree theory and especially Picard's extension to calculate the total number of real roots of Bessel functions within a predetermined interval and to isolate one of them. This procedure can be repeated for the isolation of each one of the zeros in this interval.

Once a zero is isolated, the second phase (algorithm *compute_zero*) is applied for its computation to any accuracy (subject to relative machine precision). This algorithm utilizes a modified bisection method. The only computable information required consists in the algebraic signs of the function and, consequently, it is not affected by imprecise function values. Moreover, it always converges rapidly to a zero within the initially specified region independently of the starting guess. It is also a globally convergent method, it can be applied to nondifferentiable continuous functions and does not involve derivatives or approximations of such derivatives. Furthermore, the number of iterations needed to compute a zero to a predetermined accuracy is a priori known.

The rootfinding portion of our method requires the smallest amount of function value information which is its algebraic sign. Using this and the stopping criterion (20), it computes a zero of a Bessel function within a given accuracy.

We have been able to calculate the total number of zeros by computing the value of the topological degree of (11) utilizing efficient methods such as STENGER's [22], STYNES' [23, 24], or KEARFOTT's method [10, 11] (see also [1]), which are based only on sign calculations.

For large argument x, one can use the usual asymptotic expressions for Bessel functions [28]. Moreover, if one would like to compute zeros of a combination of Bessel functions which may not be an alternating series, the van Wijngaarden transformation can be used to convert this new series into an alternating one [18]. Furthermore, very accurate function values for the Bessel functions can be obtained by employing STEED's and TEMME's methods [18], or, alternatively, by means of Coulomb functions, employing BARNETT's procedure [3, 4].

When just one root is required, the isolation portion of our method can be avoided if the function values at the endpoints of the given interval are opposite (see [27] for extensions). This is so because, in this case, the modified bisection method always converges to a zero.

Since the first derivatives of Bessel functions are available, it is at the user's disposal to apply any other rootfinding method, as for instance Newton's method, to accelerate the convergence within the predetermined interval. In such a case, though, unless the starting point is close enough to a zero and certainly away from an extremum of the function, convergence is not ensured. Thus, a few initial steps of our method should be performed so that these conditions are satisfied.

At last, for any given interval (a_k, b_k) containing a single zero of a Bessel function, Proposition 2 can provide a lower and an upper bound for this zero, theoretically as close to it as one desires, but practically as close to it as the accuracy of the calculations permits.

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Address: Prof. Dr. M. N. VRAHATIS, Dr. T. N. GRAPSA, Dr. O. RAGOS, Dr. F. A. ZAFIROPOULOS, Department of Mathematics, University of Patras, GR-26110 Patras, Greece