SHORT COMMUNICATIONS

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Locating and Computing Zeros of Airy Functions

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We deal with the localization and computation of the zeros of the Airy functions $Ai(z)$, $Bi(z)$ and their derivatives $Ai'(z)$, $Bi'(z)$. To this end a new algorithm is presented employing the notion of the topological degree of a continuous mapping for the localization portion. For the rootfinding part we describe an algorithm based on a modified bisection method which requires only the signs of function values. It is proved here how many terms of the Airy series are enough to give their correct sign. The algorithms locate and compute rapidly and accurately with certainty any zero. They have been implemented and tested. Performance information is reported. Lower and upper bounds of a zero are also proposed.

1. Introduction

The Airy functions $Ai(z)$ and $Bi(z)$ are the two linearly independent solutions to the differential equation [1, 12]:

$$\frac{d^2y}{dz^2} - zy = 0. \tag{1}$$

They were first considered by G. B. Airy (1838) in his studies of the intensity of light in the neighbourhood of a caustic. They often appear as the solutions to boundary value problems in electromagnetic theory and quantum mechanics [4].

$Ai(z)$ is encountered when forming the group invariant solutions to the heat equation [5], as well as the similarity solutions to the linearized Korteweg-de Vries equation [13].

$Ai(z)$ tends to zero for large positive z, while $Bi(z)$ increases unboundedly. The Airy functions are expressible in terms of Bessel functions as follows [1, 12]:

$$\left.\begin{aligned}
Ai(z) &= \frac{\sqrt{z}}{3}\,(I_{-1/3}(\zeta) - I_{1/3}(\zeta)),\\
Bi(z) &= \sqrt{\frac{z}{3}}\,(I_{-1/3}(\zeta) + I_{1/3}(\zeta)),\\
Ai(-z) &= \frac{\sqrt{z}}{3}\,(J_{-1/3}(\zeta) + J_{1/3}(\zeta)),\\
Bi(-z) &= \sqrt{\frac{z}{3}}\,(J_{-1/3}(\zeta) - J_{1/3}(\zeta)),
\end{aligned}\right\} \tag{2}$$

where $z > 0$ and

$$\zeta = \tfrac{2}{3} z^{3/2}. \tag{3}$$

In many cases the derivatives of the Airy functions $Ai'(z)$ and $Bi'(z)$ also appear:

$$\left.\begin{aligned}
Ai'(z) &= -\frac{z}{3}\,(I_{-2/3}(\zeta) - I_{2/3}(\zeta)),\\
Bi'(z) &= \frac{z}{\sqrt{3}}\,(I_{-2/3}(\zeta) + I_{2/3}(\zeta)),\\
Ai'(-z) &= -\frac{z}{3}\,(J_{-2/3}(\zeta) - J_{2/3}(\zeta)),\\
Bi'(-z) &= \frac{z}{\sqrt{3}}\,(J_{-2/3}(\zeta) + J_{2/3}(\zeta)).
\end{aligned}\right\} \tag{4}$$

$Ai(z)$, $Ai'(z)$ have zeros on the negative real axis only. $Bi(z)$, $Bi'(z)$ have real zeros on the negative real axis and complex zeros in the sector [1]:

$$\tfrac{1}{3} u < |\arg z| < \tfrac{1}{2} u. \tag{5}$$

The zeros of these functions are very useful for their study. On the other in general hand there are not analytical expressions for these zeros.

In the present paper we implement the concept of the topological degree [2] and especially the Kronecker-Picard theory [6, 7, 3] to find the exact number of real roots of Airy functions within a given region. Then this theory is used to give a new process for isolating one of these zeros. Subsequently, this root can be numerically computed to any accuracy (subject to relative machine precision) utilizing a proper method. Finally, we provide some numerical applications and give our conclusions.

2. The topological degree for the localization of zeros of Airy functions

Suppose that a real function $f(z)$ is defined and twice continuously differentiable in a bounded interval $[a, b]$ such that $f(a)\,f(b) \neq 0$. The topological degree of f at 0 relative to $[a, b]$ can be used to calculate the total number $\mathcal{N}^r$ of simple solutions of $f(z) = 0$ within (a, b). According to Picard's extension and Kronecker's integral representation of the topological degree, $\mathcal{N}^r$ is given by:

$$\mathcal{N}^r = -\frac{\xi}{\pi}\int_a^b \frac{f(z)\,f''(z) - f'^2(z)}{f^2(z) + \xi^2 f'^2(z)}\,dz + \frac{1}{\pi}\left[\arctan\left(\frac{\xi f'(b)}{f(b)}\right) - \arctan\left(\frac{\xi f'(a)}{f(a)}\right)\right], \tag{6}$$

where ξ is a suitable positive constant. Picard [6, 7] has explicitly shown that (6) is independent of the value of ξ (for a detailed derivation of Equation (6) see, for example, [11]).

The above mentioned method is applied in the sequel for the localization of the simple real zeros of the Airy function $Ai(z)$. In this case Relation (6) can be written as follows:

$$\mathcal{N}^r = -\frac{\xi}{\pi}\int_a^b G(z)\,dz + \frac{1}{\pi}\left[\arctan\left(\frac{\xi Ai'(b)}{Ai(b)}\right) - \arctan\left(\frac{\xi Ai'(a)}{Ai(a)}\right)\right], \tag{7}$$

where

$$G(z) = \frac{Ai(z)\,Ai''(z) - Ai'^2(z)}{Ai^2(z) + \xi^2 Ai'^2(z)}. \tag{8}$$

We describe now the respective algorithm which results in the exact number of roots of the Airy function $Ai(z)$ existing within a predetermined interval (a, b), isolates one "arbitrary" zero bisecting this interval and gives a lower bound a^* and an upper one b^* for this root. d denotes the predetermined accuracy for the integration in Relation (7).

Algorithm 1: *Localization of a real zero of the Airy function $Ai(z)$.*

Step 1. *Input* $\{a; b; \xi; \delta\}$.
Step 2. Define $G(z)$ from Relation (8).
Step 3. Define $H(a_1, a_2) = \dfrac{1}{\pi}\arctan\left(\dfrac{\xi Ai'(a_1)}{Ai(a_1)}\right) - \dfrac{1}{\pi}\arctan\left(\dfrac{\xi Ai'(a_2)}{Ai(a_2)}\right)$.
Step 4. Compute $I = -\dfrac{\xi}{\pi}\displaystyle\int_a^b G(z)\,dz$ within the accuracy δ and set $\mathrm{Int}_0 = I + H(b, a)$.

Step 5. Set $\mathcal{N}^r = \mathrm{Int}_0$.

Step 6. If $\mathrm{Int}_0 = 1$ or 0 go to Step 15.

Step 7. Set $m = (a + b)/2$ and check whether $Ai(m) = 0$; if so, set $a = m - \gamma$ and $b = m + \gamma$ (where γ is the relative machine precision) and go to Step 15; otherwise continue.

Step 8. Compute $I = -\frac{\xi}{\pi}\int_a^m G(z)\,dz$ within the accuracy δ and set $\mathrm{Int}_1 = I + H(m, a)$.

Step 9. Set $\mathrm{Int}_2 = \mathrm{Int}_0 - \mathrm{Int}_1$.

Step 10. If $\mathrm{Int}_1 = 1$, set $b = m$ and go to Step 15.

Step 11. If $\mathrm{Int}_2 = 1$, set $a = m$ and go to Step 15.

Step 12. If $\mathrm{Int}_1 = 0$, set $a = m$; go to Step 7.

Step 13. If $\mathrm{Int}_2 = 0$, set $b = m$; go to Step 7.

Step 14. $\mathrm{Int}_1 \leqq \mathrm{Int}_2$, set $b = m$, $\mathrm{Int}_0 = \mathrm{Int}_1$ and go to Step 7; otherwise set $a = m$, $\mathrm{Int}_0 = \mathrm{Int}_2$ and go to Step 7.

Step 15. Set $a^* = a$ and $b^* = b$.

Step 16. *Output* $\{a^*; b^*; \mathcal{N}^r\}$.

Remark 1: In order to localize zeros of other Airy functions, $Bi(z)$, $Ai'(z)$, $Bi'(z)$, Steps 2 and 3 of the above algorithm have to be adapted.

Remark 2: The second derivatives Ai'' and Bi'' appearing in $G(z)$ can be obtained by use of Equation (1). The third derivatives Ai''' and Bi''' involved when dealing with zeros of Ai' and Bi' can be taken by differentiating (1).

3. Computing zeros of Airy functions

A solution of $f(z) = 0$, where $f: [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function, is guaranteed to exist in the interval $[a, b]$ if $f(a)\,f(b) \leqq 0$. This is known as Bolzano's criterion and can be generalized to higher dimensions [10]. Based on this condition various rootfinding methods, as for example bisection methods, are created. Here we shall use a simplified version of the bisection method described in [8, 9]. It is reported there that, in order to compute a root of $f(z)$, the following iterative formula can be used:

$$z_{i+1} = z_i + \operatorname{sgn} f(z_0) \operatorname{sgn} f(z_i)\, h/2^{i+1}, \qquad i = 0, 1, \dots, \tag{9}$$

with $z_0 = a$, $h = b - a$. The iterations (9) converge to a root $r \in (a, b)$ if for some z_i, $i = 1, 2, \dots$, there holds:

$$\operatorname{sgn} f(z_0) \operatorname{sgn} f(z_i) = -1 .$$

The number of iterations h required to obtain an approximate root r^* such that $|r - r^*| \leqq \varepsilon$ for some $e \in (0, 1)$, is given by:

$$h = \lceil \log_2 (h\varepsilon^{-1}) \rceil ,$$

where the notation $\lceil \cdot \rceil$ refers to the smallest integer not less than the real number quoted.

Instead of the iterative formula (9) we can also use the following one:

$$z_{i+1} = z_i - \operatorname{sgn} f(z_0) \operatorname{sgn} f(z_i)\, h/2^{i+1}, \qquad i = 0, 1, \dots, \tag{10}$$

with $z_0 = b$ and $h = b - a$.

The schemes described above have all the advantages of the bisection method. Also, as it is evident from (9) and (10), the only computable information required by them is on the algebraic sign of the function f.

To determine the number of terms necessary for the derivation of the algebraic sign of the series expressions of the Airy functions we state the following theorems.

Theorem 1: *The sign of the Airy function $Ai(-z)$, $z > 0$, is the same as the sign of the summation*

$$L(z) = \sum_{m=0}^{M+N} (-1)^m c_m , \tag{11}$$

where

$$c_m = \frac{z^{3m+1/2}}{3^{2m+1/3} m!\, \Gamma(m + 4/3)} + \frac{z^{3m-1/2}}{3^{2m-1/3} m!\, \Gamma(m + 2/3)},$$

$$M = \left\lceil -\frac{5}{6} + \sqrt{\frac{z^2}{4} + \frac{1}{36}} \right\rceil,$$

and N is such that the following relation holds:

$$|c_{M+N+1}| < \left| \sum_{m=0}^{M+N} (-1)^m c_m \right| . \tag{12}$$

Proof: Let $J_\nu(z)$ be the Bessel function

$$J_\nu(z) = \sum_{m=0}^{\infty} (-1)^m a_m(n, z) ,$$

where

$$a_m(n, z) = \frac{z^{2m+\nu}}{2^{2m+\nu} m!\, \Gamma(\nu + m + 1)} .$$

Then, from Relations (2) we have that

$$Ai(-z) = \frac{\sqrt{z}}{3} \left[\sum_{m=0}^{\infty} (-1)^m \left(a_m(1/3, z) + a_m(-1/3, z)\right) \right] .$$

The sequences $a_m(1/3, z)$ and $a_m(-1/3, z)$ fulfil the Leibnitz Theorem conditions for

$$m > \left\lceil -\frac{5}{6} + \sqrt{\frac{z^2}{4} + \frac{1}{36}} \right\rceil .$$

Consequently, the Airy function $Ai(-z)$ satisfies the conditions of the Leibnitz theorem. So, the series

$$R(z) = \sum_{m=M}^{\infty} (-1)^m c_m(z) ,$$

where

$$c_m(z) = a_m(1/3, z) + a_m(-1/3, z) ,$$

is an alternating series for which the three conditions of the Alternating Series Estimation Theorem hold, thus ensuring that its $(M + N + 1)$th term is larger than the absolute value of the remainder of $R(z)$. Furthermore, the signs of the $(M + N + 1)$th term and of the remainder of $R(z)$ are the same. Now, by Assumption (12), the sign of the Airy function $Ai(-z)$ is the same as the corresponding sign of $L(z)$. Thus the theorem is proved. □

Working in the same way we are able to prove the following theorems.

Theorem 2: *The sign of the Airy function $Bi(-z)$, $z > 0$, is the same as the sign of the summation*

$$L(z) = \sum_{m=0}^{M+N} (-1)^m c_m , \tag{13}$$

where

$$c_m = \frac{z^{3m}}{3^{2m+1/6} m!\, \Gamma(m + 2/3)} - \frac{z^{3m+1}}{3^{2m+5/6} m!\, \Gamma(m + 4/3)},$$

$$M = \left\lceil -\frac{5}{6} + \sqrt{\frac{z^2}{4} + \frac{1}{36}} \right\rceil,$$

and N is such that the following relation holds:

$$|c_{M+N+1}| < \left| \sum_{m=0}^{M+N} (-1)^m c_m \right| . \tag{14}$$

Theorem 3: *The sign of the derivative $Ai'(-z)$, $z > 0$, is the same as the sign of the summation*

$$L(z) = \sum_{m=0}^{M+N} (-1)^m c_m , \tag{15}$$

where

$$c_m = \frac{z^{3m+2}}{3^{2m+5/3} m!\, \Gamma(m+5/3)} - \frac{z^{3m}}{3^{2m+1/3} m!\, \Gamma(m+1/3)},$$

$$M = \left\lceil -\frac{2}{3} + \sqrt{\frac{z^2}{4} + \frac{1}{9}} \right\rceil,$$

and N is such that the following relation holds:

$$|c_{M+N+1}| < \left| \sum_{m=0}^{M+N} (-1)^m c_m \right|. \tag{16}$$

Theorem 4: *The sign of the derivative $Bi'(-z)$, $z > 0$, is the same as the sign of the summation*

$$L(z) = \sum_{m=0}^{M+N} (-1)^m c_m, \tag{17}$$

where

$$c_m = \frac{z^{3m+2}}{3^{2m+5/6} m!\, \Gamma(m+5/3)} - \frac{z^{3m}}{3^{2m-1/6} m!\, \Gamma(m+1/3)},$$

$$M = \left\lceil -\frac{2}{3} + \sqrt{\frac{z^2}{4} + \frac{1}{9}} \right\rceil,$$

and N is such that the following relation holds:

$$|c_{M+N+1}| < \left| \sum_{m=0}^{M+N} (-1)^m c_m \right|. \tag{18}$$

Next we give the description of an appropriate algorithm for the computation of a real zero of the Airy function $Ai(z)$. a_v and b_v are the left and right bounds, respectively, and e is the predetermined accuracy.

Algorithm 2: *Computation of a real zero of the Airy function $Ai(z)$.*

Step 1. *Input* $\{a_v; b_v; e\}$.
Step 2. Set $h_v = b_v - a_v$.
Step 3. Set $h = \lceil \log_2 (h_v e^{-1}) \rceil$.
Step 4. Set $z_0 = a_v$.
Step 5. Set $M = \left\lceil -\frac{5}{6} + \sqrt{\frac{z_0^2}{4} + \frac{1}{36}} \right\rceil$.
Step 6. Set $S = \sum_{m=0}^{M} (-1)^m \left[\frac{z_0^{3m+1/2}}{3^{2m+1/3} m!\, \Gamma(m+4/3)} + \frac{z_0^{3m-1/2}}{3^{2m-1/3} m!\, \Gamma(m+2/3)} \right]$.
Step 7. Set $m = M + 1$.
Step 8. Set $T = (-1)^m \left[\frac{z_0^{3m+1/2}}{3^{2m+1/3} m!\, \Gamma(m+4/3)} + \frac{z_0^{3m-1/2}}{3^{2m-1/3} m!\, \Gamma(m+2/3)} \right]$.
Step 9. If $|T| \geqq |S|$, then set $S = S + T$, replace m by $m + 1$ and return to Step 8; otherwise set $s_0 = \operatorname{sgn}(S + T)$ and go to the next step.
Step 10. Set $i = -1$.
Step 11. If $i \leqq h$, replace i by $i + 1$ and go to the next step; otherwise, go to Step 18.
Step 12. Set $M = \left\lceil -\frac{5}{6} + \sqrt{\frac{z_i^2}{4} + \frac{1}{36}} \right\rceil$.
Step 13. Set $S = \sum_{m=0}^{M} (-1)^m \left[\frac{z_i^{3m+1/2}}{3^{2m+1/3} m!\, \Gamma(m+4/3)} + \frac{z_i^{3m-1/2}}{3^{2m-1/3} m!\, \Gamma(m+2/3)} \right]$.
Step 14. Set $m = M + 1$.
Step 15. Set $T = (-1)^m \left[\frac{z_i^{3m+1/2}}{3^{2m+1/3} m!\, \Gamma(m+4/3)} + \frac{z_i^{3m-1/2}}{3^{2m-1/3} m!\, \Gamma(m+2/3)} \right]$.
Step 16. If $|T| \geqq |S|$, then set $S = S + T$, replace m by $m + 1$ and return to Step 15; otherwise set $s_i = \operatorname{sgn}(S + T)$ and go to the next step.
Step 17. Set $z_{i+1} = z_i + s_0 s_i h_v / 2^{i+1}$ and return to Step 11.
Step 18. *Output* $\{z_i\}$.

Remark 3: In order to obtain zeros of the other Airy functions, Steps 6, 8, 13, and 15 of the above algorithm have to be changed accordingly.

Table I: Several runs of Algorithm 1

a	b	$\mathcal{N}$ of Ai	$\mathcal{N}$ of Bi	$\mathcal{N}$ of Ai'	$\mathcal{N}$ of Bi'
− 5.	0.	2	3	3	2
− 10.	− 5.	4	4	4	4
− 15.	−10.	6	6	6	6
− 20.	−15.	7	6	6	7
− 25.	−20.	7	8	8	7
− 30.	−25.	9	8	8	9
− 35.	−30.	9	9	9	9
− 40.	−35.	9	10	10	9
− 45.	−40.	11	10	10	11
− 50.	−45.	11	11	11	11
− 55.	−50.	11	12	12	11
− 60.	−55.	12	12	12	12
− 65.	−60.	13	12	12	13
− 70.	−65.	13	14	14	13
− 75.	−70.	14	13	13	14
− 80.	−75.	14	14	14	14
− 85.	−80.	14	15	15	14
− 90.	−85.	15	14	14	15
− 95.	−90.	15	16	16	15
−100.	−95.	16	15	15	16

Table II: Several runs of Algorithm 2

i	r_i of Ai	r_i of Bi	r_i of Ai'	r_i of Bi'
1	− 2.3381074101	− 1.1737132222	− 1.0187929723	− 2.2944396830
2	− 4.0879494438	− 3.2710933025	− 3.2481975825	− 4.0731550894
3	− 5.5205598278	− 4.8307378414	− 4.8200992115	− 5.5123957299
4	− 6.7867080898	− 6.1698521281	− 6.1633073559	− 6.7812944462
5	− 7.9441335869	− 7.3767620791	− 7.3721772553	− 7.9401786894
6	− 9.0226508531	− 8.4919488463	− 8.4884867342	− 9.0195833590
7	−10.0401743414	− 9.5381943791	− 9.5354490526	−10.0376963351
8	−11.0085243036	−10.5299135065	−10.5276603971	−11.0064626679
9	−11.9360155631	−11.4769535511	−11.4750566337	−11.9342616452
10	−12.8287767527	−12.3864171384	−12.3847883720	−12.8272583093
11	−13.6914890350	−13.2636395228	−13.2622189618	−13.6901558270
12	−14.5278299516	−14.1127568089	−14.1115019706	−14.5266457636
13	−15.3407551358	−14.9370574120	−14.9359371969	−15.3396930824
14	−16.1326851568	−15.7392103510	−15.7382013738	−16.1317247825
15	−16.9056339973	−16.5214195505	−16.5205038256	−16.9047594120
16	−17.6613001056	−17.2855316244	−17.2846950504	−17.6604987433
17	−18.4011325991	−18.0331132871	−18.0323446226	−18.4003943673
18	−19.1263804741	−18.7655082843	−18.7647984378	−19.1256971566
19	−19.8381298916	−19.4838801329	−19.4832216567	−19.8374947186
20	−20.5373329075	−20.1892447853	−20.1886315096	−20.5367402416
21	−21.2248299435	−20.8824959941	−20.8819227556	−21.2242750450
22	−21.9013675955	−21.5644252846	−21.5638877233	−21.9008464453
23	−22.5676129174	−22.2357378817	−22.2352322855	−22.5671220806
24	−23.2241650010	−22.8970655541	−22.8965887390	−23.2237015213
25	−23.8715644554	−23.5489770795	−23.5485262961	−23.8711257718
26	−24.5103012365	−24.1919868505	−24.1915597096	−24.5098851171
27	−25.1408211660	−24.8265620120	−24.8261564260	−25.1404256555
28	−25.7635314009	−25.4531284270	−25.4527425619	−25.7631547770
29	−26.3788050520	−26.0720756983	−26.0717079353	−26.3784457913
30	−26.9869851115	−26.6837614250	−26.6834103284	−26.9866418599
31	−27.5883878098	−27.2885148300	−27.2881791216	−27.5880593593
32	−28.1833055025	−27.8866398716	−27.8863184089	−28.1829907714
33	−28.7720091651	−28.4784179256	−28.4781096832	−28.7717071810
34	−29.3547505587	−29.0641101077	−29.0638141628	−29.3544604447
35	−29.9317641190	−29.6439592958	−29.6436748147	−29.9314850821
36	−30.5032686113	−30.2181918969	−30.2179181246	−30.5029999320
37	−31.0694685851	−30.7870193978	−30.7867556481	−31.0692096088
38	−31.6305556579	−31.3506397311	−31.3503853792	−31.6303057877
39	−32.1867096528	−31.9092384835	−31.9089929585	−32.1864683427
40	−32.7380996089	−32.4629899667	−32.4627527463	−32.7378663585

4. Applications and concluding remarks

We have tested our algorithms on Ai, Bi, Ai' and Bi' using several random intervals (a, b). Our experience is that our methods behave predictably and accurately. The utilized input values of ξ and δ of Algorithm 1 and ε of Algorithm 2 were 10^{-2}, 10^{-3}, and 10^{-15}, respectively. Some of the obtained results are exhibited in Tables I and II. In the first table we give the total number of roots $\mathcal{N}^r$ of the above mentioned Airy functions existing within several intervals. In the second table we display, giving ten decimal digits, the "first" 40 roots r_i of these functions which were computed by Algorithm 2.

As mentioned above, Algorithm 1 localizes only one of the roots inside (a, b). If someone needs to isolate the rest of the zeros, one has to repeat the whole procedure for the remaining intervals (a, a^*) and (b^*, b). Furthermore, if the total number of roots in the interval (a, b) is odd, then the user is able to apply the bisection method in order to compute one of them within an accuracy ε and, of course, to localize this zero in a small interval with length 2ε (for details, see [11]).

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On the R-Order of a Generalization of Single-Step Weierstrass Type Methods

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This note deals with the R-order of convergence of the Weierstrass type single-step methods for the simultaneous determination of all simple polynomial roots.

1. Introduction

Consider a polynomial of degree $n > 3$,

$$f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 = \prod_{i=1}^{n} (x - x_i), \tag{1.1}$$

with only simple real or complex roots $x_1, x_2, \dots, x_n$.

The polynomial equation $f(x) = 0$ leads to the following fixed point equation,

$$x_i = x - \frac{f(x)}{\prod\limits_{j=1, j\neq i}^{n} (x - x_j)}, \qquad i = 1, 2, \dots, n. \tag{1.2}$$

Suppose that $x_1^k, x_2^k, \dots, x_n^k$ are distinct, reasonably close approximations to the simple roots $x_1, x_2, \dots, x_n$ of (1.1) at the k-th iterative step. Then the equation (1.2) gives raise to the following iterative method introduced by Weierstrass [8] for the simultaneous determination of all simple roots of (1.1):

$$x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{\prod\limits_{j=1, j\neq i}^{n} (x_i^k - x_j^k)}, \qquad i = 1, 2, \dots, n; \qquad k = 0, 1, \dots. \tag{1.3}$$

The Weierstrass method (1.3) was subject of investigations by several authors and a survey on its properties can be found in [9]. It has been shown in [5] that the method (1.3) is the Newton method applied to the Viéte system of n equations in $x_1, \dots, x_n$, obtained by equating to a_i the coefficients of x^i for $i = 0, \dots, n-1$ in the expansion of $\prod_{j=1}^{n} (x - x_j)$ as a polynomial in x (see also [9]). The iteration (1.3) has local quadratic convergence if it starts close enough to the roots $x_1, \dots, x_n$ of (1.1).

The convergence order can be increased by calculating the new approximations x_i^{k+1}, $1 \leqq i \leqq n$, in (1.3) using the already calculated approximations $x_1^{k+1}, \dots, x_{i-1}^{k+1}$ (the so-called Gauss-Seidel approach). Namely, the Gauss-Seidel procedure applied to the method (1.3) leads to the iteration

$$x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{\prod\limits_{j=1}^{i-1} (x_i^k - x_j^{k+1}) \prod\limits_{j=i+1}^{n} (x_i^k - x_j^k)},$$
$$i = 1, 2, \dots, n; \qquad k = 0, 1, \dots, \tag{1.4}$$

for the simultaneous determination of all simple roots of a polynomial f.

The Gauss-Seidel method (1.4) was analyzed by Alefeld and Herzberger in the case of real roots of a given polynomial f [1]. Let $h_i^k = x_i^k - x_i$, $i = 1, 2, \dots, n$, $k = 0, 1, \dots$. Then, the following estimation can be verified [1]:

$$|h_i^{k+1}| \leqq C\, |h_i^k| \left(\sum_{j=1}^{i-1} |h_j^{k+1}| + \sum_{j=i+1}^{n} |h_j^k| \right). \tag{1.5}$$

Using (1.5), Alefeld and Herzberger proved that the R-order of convergence of procedure (1.4) is at least $1 + \sigma_n$, where σ_n is the unique positive solution of the equation $\sigma^n - \sigma - 1 = 0$.

For simultaneously determining all roots of a given polynomial (1.1) the following total-step method (TSM),

$$\left.\begin{aligned} & x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{\prod\limits_{j=1, j\neq i}^{n} (x_i^k - x_j^k - \Delta_j^{R,k})}, \qquad 1 \leqq i \leqq n, \\ & \Delta_i^{p,k} = -\frac{f(x_i^k)}{\prod\limits_{s=1, s\neq i}^{n} (x_i^k - x_s^k - \Delta_s^{p-1,k})}, \qquad \Delta_i^{0,k} = 0, \\ & i = 1, 2, \dots, n; \quad p = 1, 2, \dots, R; \quad k = 0, 1, \dots, \end{aligned}\right\} \tag{1.6}$$

was proposed by Andreev and Kjurkchiev in [6]. For a fixed integer $R > 0$ the method (1.6) is a modification of the Weierstrass method (1.3) with raised order of convergence.

Let x_i^0, $i = 1, \dots, n$, be the initial approximations for the roots such that all simple roots x_i, $i = 1, \dots, n$, are contained in the discs centered at x_i^0 with radii p, i.e. $|x_i^0 - x_i| \leqq p$ for $i = 1, \dots, n$. Then, the following assertion for the order of convergence of (1.6) was given in [3] (see also [6]).