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A Procedure to Compute the Fixed Points and Visualize the Orbits of a 2D Map

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Abstract

The accurate computation of periodic orbits of nonlinear mappings and the precise knowledge of their properties are very important for studying the behaviour of many dynamical systems of physical interest. In this paper we describe a new software package, called **giotto**, which visualizes orbits of nonlinear mappings of dynamical systems and computes all the periodic orbits (stable and unstable) of any period and accuracy which occur in the study of beam dynamics in particle accelerators.

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1. Introduction

The nonlinear betatronic motion in a large hadron accelerator is conveniently described by the one–turn map on a given reference section. The geometry of orbits in phase space and their stability properties are consequently analyzed by iterating the map (tracking) or by constructing its normal form[2–15].

An effective way of tracking the orbits is provided by interactive graphics procedures on a workstation, since the regions where the most interesting or intriguing features appear can be localized and explored to the desired accuracy. In this respect the knowledge of fixed points is relevant since they determine the local features of the orbits; indeed the Taylor expansion of the map at a fixed point allows a local normal form analysis[2–15]. Implementing a viewer of 4D maps combined with a fixed points finder and normal forms algorithms is a long range program. Here we describe an efficient algorithm to compute the fixed points and the interactive procedure giotto to visualize the orbits of a 2D mapping. The inclusion of the interactive computation of normal forms is in progress and the extension to 4D mapping under investigation.

Fixed points

Given a two dimensional map[16,22,23]

$$F : \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \tag{1.1}$$

we say that $\mathrm{x} = (x,y)$ is a fixed point of F if $F(\mathrm{x}) = \mathrm{x}$ and a fixed point of order q (or a periodic orbit of period q) if

$$\mathrm{x} = F^q(\mathrm{x}) \equiv F(F(\ldots(F(\mathrm{x}))\ldots)) \qquad (q \text{ times}) \tag{1.2}$$

In general analytic expressions are available only if the map is a polynomial of low degree and the period is low. On the other hand it is difficult to find in the literature efficient methods for computing high period orbits if the map is not decomposable into involutions[17,22,23]. Also, traditional iterative schemes such as Newton's method and related classes of algorithms[28,ch. 7] often fail since they converge to a fixed point almost independently of the initial guess, while there exist several fixed points, close to each other, which are all desirable for the applications[1]. Moreover these methods are affected by the mapping evaluations taking large values in neighborhoods of hyperbolic fixed points. Finally, in general, these methods often fail due to the nonexistence of derivatives or poorly behaved partial derivatives[28].

In this report we describe an efficient algorithm based on the topological degree theory to provide a criterion for the existence of a fixed point within a given region. More specifically the method constructs a polyhedron in such a way that the value of the topological degree of an iterate of the mapping, relative to this polyhedron, be ± 1, which implies the existence of a fixed point within this polyhedron. Then it repeatedly subdivides this polyhedron in such a way that the new refined polyhedron also retains the property of the existence of a fixed point within its interior, without any computation of the topological degree. These subdivisions take place iteratively, until a fixed point is computed to a predetermined

accuracy. When a fixed point is computed, the method iterates the mapping to obtain all the fixed points of the same period to the same accuracy.
This algorithm becomes especially significant for the computation of high period fixed points (elliptic and hyperbolic) where other more traditional approaches (like Newton's method, etc.) cannot easily distinguish among closely neighboring fixed points.
Here we shall illustrate this method on Hénon's quadratic mapping[23,22,16]

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathsf{R}(\alpha) \begin{pmatrix} x \\ y + f(x) \end{pmatrix}, \tag{1.3}$$

where

$$f(x) = -x^2, \qquad \mathsf{R}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

to compute fixed points for given values of α and various periods.
The criterion for the existence of a fixed point using the characteristic polyhedron and details for its construction can be found in [31,32,35]; here we point out that the initial guess of such a polyhedron is easier and fast if an interactive viewer is available:

giotto is an interactive viewer of the orbits of an area preserving map. It has been designed in such a way that the user has only to provide a FORTRAN or C function to compute the map and includes, in its library, the algorithm for the search of fixed points.
The viewer allows magnifications of any given region in phase space so that the fixed points of interest can be localized providing a good initial guess of the characteristic polyhedron.
The use of giotto is simple: the beginning of a session is guided by a series of questions to provide the basic parameters and the graphic session is menu oriented for all the desired operations such as changing colour, magnifying a region, erasing orbits, searching fixed points, writing a PS file to dump the screen for later printing.

The plan of this report is the following: in section 2 we define the topological degree and illustrate the construction of the characteristic polyhedron, in section 3 we describe the bisection method to refine the result, in section 4 we analyze the application of the method to the 2D Hénon map, in section 5 we describe the viewer giotto and in section 6 we provide an example of its use.

2. The topological degree.

In this section we shall implement topological degree theory to give a criterion for the existence of a fixed point within a given region. This criterion is based on the construction of a "characteristic polyhedron" within a scaled translation of the unit cube. So, in the sequel, by reviewing the concept of a characteristic polyhedron we briefly present a procedure for the construction of this polyhedron. The theoretical development of the concepts employed here can be found in[31,32,35].
Bisection methods for finding roots of systems of equations depend on a "criterion" whose result can guarantee that a root will lie within a given region; then this region can be

subdivided in such a way that the result of the criterion can be again applied within the new refined region.
In one dimension, this criterion consists in examining the signs of the function evaluations at the endpoints of the given region. Specifically, if we desire to locate a root of the equation $f(x) = 0$ in the interval (a, b), where $f : (a, b) \subset \mathbf{R} \rightarrow \mathbf{R}$ is continuous, we can examine whether the following relation is fulfilled:

$$\text{sgn } f(\alpha)\text{sgn } f(\beta) < 0 \tag{2.1}$$

where sgn is the well known sign function with values

$$\text{sgn } \psi = \begin{cases} -1 & \psi < 0 \\ 0 & \psi = 0 \\ 1 & \psi > 0 \end{cases} \tag{2.2}$$

and if so, then we certainly know that there is at least one solution within (α, β) (see figure 1). The above criterion is known as Bolgano's criterion.

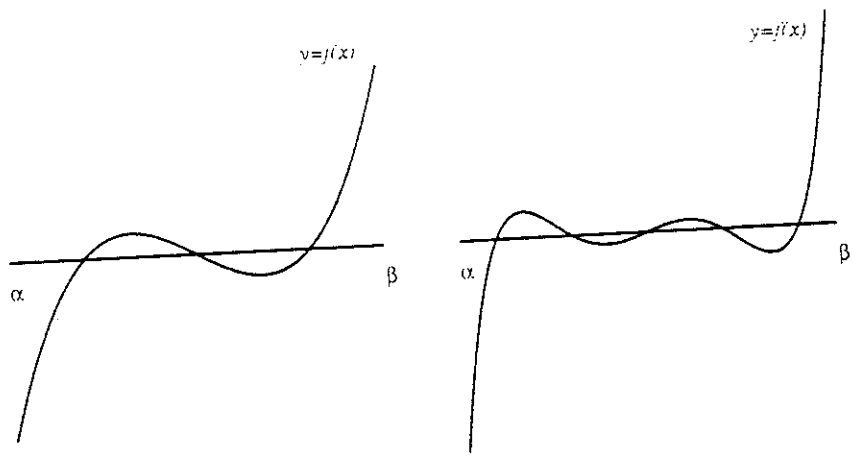


Fig.1:Bolgano's criterion for the existence of at least one root of $f(x) = 0$ within (α, β)

Instead of Bolgano's criterion we can also use the following criterion

$$\deg[f, (a, b), 0] = \frac{1}{2}\{\text{sgn } f(b) - \text{sgn } f(a)\}. \tag{2.3}$$

Now if the value of $\deg[f, (a, b), 0]$ is not zero we know with certainty that there exists at least one solution in (a, b).
The $\deg[f, (a, b), 0]$ is called *topological degree* of f at zero, relative to (a, b)[28]. Note that if the value of $\deg[f, (a, b), 0]$ is not zero, then the Bolgano's criterion is fulfilled. Also the value of $\deg[f, (a, b), 0]$ gives additional information concerning the behaviour of the roots of $f(x) = 0$ in (a, b), relative to the slopes of $f(x)$. For example, if $\deg[f, (a, b), 0] = 1$, which means that $f(b) > 0$ and $f(a) < 0$, then the number of roots at points where $f(x)$

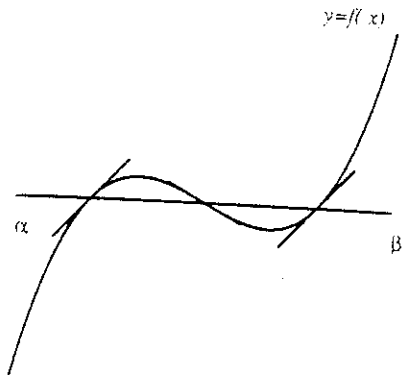


Fig.2:The slopes of $f(x)$ at roots

has a positive slope exceeds by one the number of roots at points where $f(x)$ has a negative slope[25] (see figure 2).
Now, using the value of the topological degree, we are able to calculate a root of $f(x) = 0$ by bisecting the interval (a, b). So we subdivide (a, b) into two intervals $(a, c][c, b)$ where $c = (a + b)/2$ is the midpoint of (a, b) and we keep the subinterval for which the value of topological degree is not zero, relative to itself. In this way we keep the existence of at least one root within a smaller interval. Now we can continue this procedure to approximate a root until the endpoints of the final collected subinterval differ from each other less than a prefixed amount.
This method (called *bisection method*) can be written as follows[19,20,21,31,32,34,35]:

$$x_{n+1} = x_n + \text{sgn } f(a)\text{sgn } f(x_n)(b-a)/2^{n+1}, \qquad x_0 = a, \qquad n = 0, 1, \ldots \tag{2.4}$$

A straightforward extension of Bolgano's criterion to two dimensions is to consider a box such that the function $f_1(x, y)$ may possess opposite signs at opposite sides and the function $f_2(x, y)$ also has opposite signs at the remaining sides of the square (see figure 3)
Then if this criterion is satisfied there is at least one solution of the system

$$\begin{aligned} f_1(x, y) &= 0 \\ f_2(x, y) &= 0 \end{aligned} \tag{2.5}$$

within the given box. This square, which is known as Miranda's box[27,33] can be generalized in the case of higher dimensions to provide the criterion of existence. The meaning of Miranda's box is evident if we consider a sufficiently small neighborhood $\mathcal{A}$ of a point $\mathbf{x}_*$ where (2.5) is satisfied with a non-vanishing jacobian, so that no other solution of (2.5) exists in $\mathcal{A}$. In this case $\xi_1 = f_1$ and $\xi_2 = f_2$ define a system of curvilinear coordinates, with origin at $\mathbf{x}_*$, subdividing $\mathcal{A}$ into four quadrants whose boundaries are the curves $f_1 = 0$, $f_2 = 0$. Joining any point in the first ($f_1 > 0, f_2 > 0$), second ($f_1 > 0, f_2 < 0$), third ($f_1 < 0, f_2 < 0$) and fourth ($f_1 < 0, f_2 > 0$) quadrant one obtains a Miranda's box which obviously contains the origin. However, although Miranda's box gives a criterion

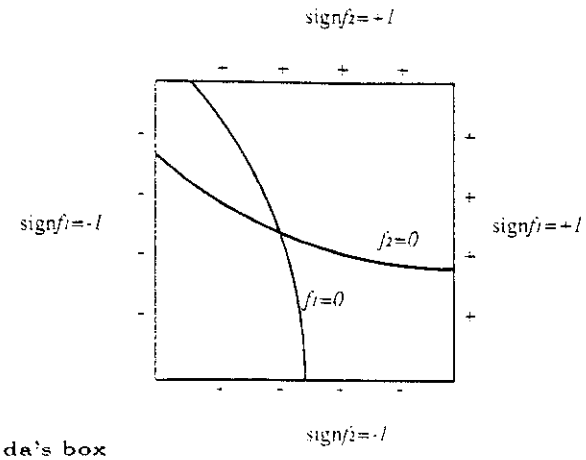


Fig.3:Miranda's box

with certainty for the existence of a root, its construction is very often impossible and cannot be applied for the computation. In overcoming these difficulties we give a "suitable" polyhedron whose construction guarantees the existence of at least one root in its interior. Before presenting the concept of this polyhedron, let us give a generalization of the topological degree to two dimensions, on which various generalized methods of bisection are based. According to these methods one establishes the existence of at least one root of the system (2.5) where $F = (f_1, f_2) : \mathcal{D} \subset \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is continuous on the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ such that $F(\mathbf{x}) \neq 0$ for $\mathbf{x}$ on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$, by computing the value of the topological degree of F at $\Theta = (0,0)$ relative to $\mathcal{D}$. This is denoted by $\deg[F, \mathcal{D}, \Theta]$ and can be defined by the following sum

$$\deg[F, \mathcal{D}, \Theta] = \sum_{\mathbf{x} \in F^{-1}(\Theta)} \text{sgn } \det J_F(\mathbf{x}) \tag{2.6}$$

where $\det J_F(\mathbf{x})$ indicates the determinant of the jacobian matrix. Now, if a nonzero value of $\deg[F, \mathcal{D}, \Theta]$ is obtained, then by Kronecker's existence theorem[28,p.161] it follows that there is at least one root of the system within $\mathcal{D}$. On the other hand, if $\deg[F, \mathcal{D}, \Theta] = 0$, no conclusion can be drawn because more information about F is needed[28].
However, although the nonzero value of $\deg[F, \mathcal{D}, \Theta]$ plays an important role in the existence of a root, its exact value is useless, since it does not give any additional information about the existence of the root of system (2.5). Moreover the computation of $\deg[F, \mathcal{D}, \Theta]$ is a time–consuming procedure and cannot be accurately achieved unless the modulus of continuity of F on $\mathcal{D}$ is known. For more details about degree theory we refer the reader to[28].
In what follows, we give the concept of a characteristic polyhedron. To define it, we have to give some other tools. Consider a number i with n binary digits

$$0 \leq i = b_0^i + 2b_1^i + \cdots + 2^{n-1} b_{n-1}^i \leq 2^n - 1 \tag{2.7}$$

where $b_j^i = 0, 1$ and the matrices $\mathcal{M}_n$ defined by

$$[\mathcal{M}_n]_{i+1,j} = 2b_j^i - 1 \tag{2.8}$$

For $n = 1$ and $n = 2$ we have respectively

$$\mathcal{M}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathcal{M}_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{2.9}$$

If $n = 1$ consider the segment $[x_1, x_2]$ and evaluate the sign of $f(x)$ at the end points: if the matrix

$$S(f; x_1, x_2) = \begin{bmatrix} \operatorname{sgn} f(x_1) \\ \operatorname{sgn} f(x_2) \end{bmatrix} \tag{2.10}$$

agrees with $\mathcal{M}_1$, up to a permutation of the rows, then we say that $[x_1, x_2]$ is a characteristic polyhedron.
For $n = 2$ consider four points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$, vertices of a quadrilateral box (polyhedron) as shown in figure 4. Letting $F = (f_1, f_2)$ be a map of the plane we consider the matrix of signs of the functions f_1, f_2 at the vertices of the polyhedron defined by

$$S(F; X_1X_2X_3X_4) = \begin{bmatrix} \operatorname{sgn} F(X_1) \\ \operatorname{sgn} F(X_2) \\ \operatorname{sgn} F(X_3) \\ \operatorname{sgn} F(X_4) \end{bmatrix} = \begin{bmatrix} \operatorname{sgn} f_1(X_1) & \operatorname{sgn} f_2(X_1) \\ \operatorname{sgn} f_1(X_2) & \operatorname{sgn} f_2(X_2) \\ \operatorname{sgn} f_1(X_3) & \operatorname{sgn} f_2(X_3) \\ \operatorname{sgn} f_1(X_4) & \operatorname{sgn} f_2(X_4) \end{bmatrix} \tag{2.11}$$

The polyhedron $X_1X_2X_3X_4$ is called a characteristic polyhedron relative to F if the matrix of signs $S(F; X_1X_2X_3X_4)$ is identical with the 2-complete matrix $\mathcal{M}_2$. For example the polyhedron in figure 4 is characteristic, while the polyhedron in figure 5 isn't.

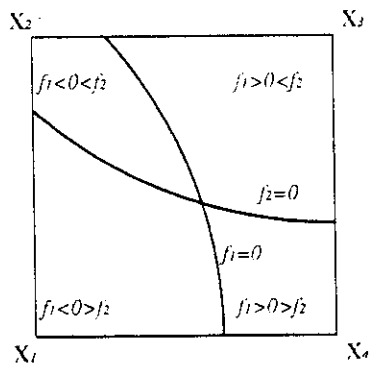


Fig.4:A characteristic polyhedron

The polyhedron $ABCD$ in figure 5 is not characteristic, since the entry $(-1, 1)$ of $\mathcal{M}_2$ is missing. Note that Miranda's box in figure 3 is a characteristic polyhedron.
Now if the boundary of $AECD$ is "sufficiently refined"[26,35] we know with certainty that there exists at least one solution within $AECD$, since the absolute value of the topological degree relative to a characteristic polyhedron is always equal to ± 1[35].

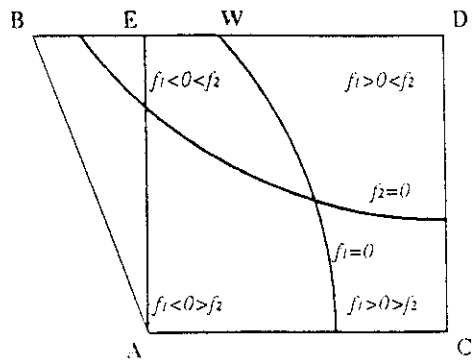


Fig.5: $ABCD$ **is not characteristic;** $AECD$ **is.**

The construction of a characteristic polyhedron starts with an arbitrary rectangle $ABCD$ which can be defined by the vertex $A = (x^0, y^0)$ and two stepsizes $h_1 = C - A$ and $h_2 = B - A$. Thus, in this case, the vertices of the rectangle $ABCD$ are:

$$A = (x^0, y^0), B = (x^0, y^0 + h_2), C = (x^0 + h_1, y^0), D = (x^0 + h_1, y^0 + h_2). \tag{2.12}$$

Now to construct a characteristic polyhedron we compare the matrix of signs $S(F; ABCD)$ with the matrix $\mathcal{M}_2$. If they are identical, then $ABCD$ is a characteristic polyhedron; otherwise we have to find out points $\mathbf{x}^*$ in $\mathbf{R}^2$ such that their vectors of signs produce the rows of $\mathcal{M}_2$ that are missing in $S(F; ABCD)$. In the case of figure 5 we observe that the row $(-1, 1)$ is missing in $S(F; ABCD)$ so we need a point $\mathbf{x}^* = (x^*, y^*)$ such that $(\operatorname{sgn} f_1(\mathbf{x}^*), \operatorname{sgn} f_2(\mathbf{x}^*)) = (-1, 1)$ which, in the case of the example of figure 5, is the point E. To find a point E we can easily compute the point W which is the root of $f_1 = 0$, lying on the edge BD. So, by holding the second component fixed and equal to $y^0 + h_2$, we can solve the one-dimensional equation

$$f_1(r, y^0 + h_2) = 0 \tag{2.13}$$

for r in the interval (B, D). Suppose that r^* is such a solution; then the point W is given by

$$W = (r^*, y^0 + h_2). \tag{2.14}$$

Now, if we change the first component of W, say

$$W = (r^* \pm \delta, y^0 + h_2) \tag{2.15}$$

by an arbitrary amount δ, we are in a position to compute a point E for the construction of a characteristic polyhedron. We can use any one of the well known one–dimensional methods to solve equation (2.15); here we use the traditional one–dimensional bisection method since frequently the edges of $ABCD$ are very long and also since a few significant digits for the computation of the root of the equation (2.15) are required. Moreover this

method has an additional advantage, since the minimum number of iterations required for obtaining an approximate root, with accuracy ϵ in some interval with length h, is known and given by

$$\mu = \lceil \log(h\epsilon^{-1}) \rceil \tag{2.16}$$

(here the notation $\lceil \cdot \rceil$ refers to the smallest integer not less than the real number quoted).

3. The bisection method.

In this section we briefly describe our method for the accurate computation of a single fixed point. This method is particularly useful in cases where the period of the fixed point is very high, since it always converges within the initial specified region. Moreover, this algorithm is very efficient since the only computable information that is required is the algebraic signs of the components of the mapping. Thus it is not affected by the mapping evaluations taking large values in neighborhoods of unstable fixed points. The theoretical development of the method can be found in[31,35].

This method is based on the refinement of a characteristic polyhedron and is called *characteristic bisection*. Specifically, the method bisects a characteristic polyhedron in such a way that the new refined polyhedron is also a characteristic one. To do this the method computes the midpoint of an edge of the characteristic polyhedron and replaces with such point that vertex of the polyhedron for which the vectors of their signs are identical. For example, as we see in figure 6, one can subdivide

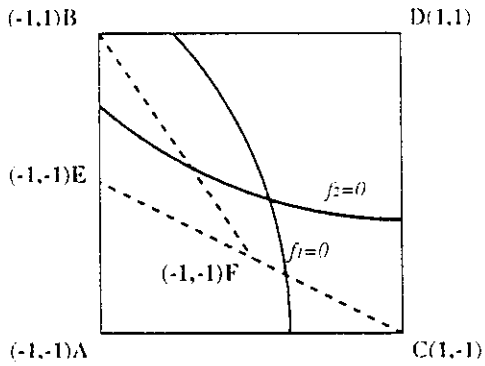


Fig.6:Characteristic bisections applied to $ABCD$. The polyhedra $ABCD$, $EBCD$ and $FBCD$ are characteristic

the polyhedron $ABCD$ in the following way: starting from the edge CD we find its midpoint E and calculate its vector of signs, which in this case is $(-1,1)$. So the vertex C is replaced by E in such a way that the new refined polyhedron $EBCD$ is also a characteristic one. Using the same procedure we are able to refine this polyhedron again by considering the midpoint F of EC and checking the vector of signs of this point. In this case its vector of signs is $(-1,1)$, hence the vertex E can be replaced by the vertex F, so that the new refined polyhedron $FBCD$ is still a characteristic one. This procedure continues until the

midpoint of the longest diagonal of a refined polyhedron approximates the root within a predetermined accuracy. Details of the above method can be found in[31,35].
The number of characteristic bisections of a characteristic polyhedron with length of its longest edge δ that is required for obtaining a new refined characteristic polyhedron with length of the longest edge smaller than ϵ is given by

$$\nu = \lceil \log_2(\delta\epsilon^{-1}) \rceil \tag{3.1}$$

(see [22] for a proof).

4. Fixed points of 2D Hénon's map.

We now give a procedure for computing all the fixed points of powers of a nonlinear mapping to any predetermined accuracy by applying the method of the previous section. The method is illustrated here on the following system of 2 equations in 2 unknowns:

$$\mathsf{T}: \begin{aligned} x_{i-1} &= x_i \cos\alpha - (y_i - x_i^2)\sin\alpha \\ y_{i-1} &= x_i \sin\alpha - (y_i - x_i^2)\cos\alpha \end{aligned} \tag{4.1}$$

with $x_n = x_0$, $y_n = y_0$ and α constant. The mapping T, expressed by the above system of equations, is known as Hénon's mapping and can be written as follows:

$$\mathsf{T} = \mathsf{R} \cdot \mathsf{S} \tag{4.2}$$

where S is a shearing parallel to the y axis:

$$x_i' = x_i, \qquad y_i' = y_i + x_i^2 \tag{4.3}$$

and R is a rotation by the angle α:

$$\begin{aligned} x_{i+1} &= x_i' \cos\alpha - y_i' \sin\alpha \\ y_{i+1} &= x_i' \sin\alpha + y_i' \cos\alpha \end{aligned} \tag{4.4}$$

A point $\mathbf{x}^*$ is called a periodic fixed point of T, with period P, $P \in N$ if $\mathsf{T}^P(\mathbf{x}^*) = \mathbf{x}^*$[29]. Such periodic orbits are identified by their rotation number $\sigma = \nu/2\pi = m_1/m_2$ where ν is the frequency of the orbit and m_1, m_2 are two positive integers[2]ch.7
Now to compute a fixed point of a particular period P, we consider the following operator

$$\mathsf{F} = \mathsf{T}^P - \mathsf{I}, \tag{4.5}$$

where I is the identity mapping, and solve the following system of equations

$$\mathsf{F}(x) = \Theta = (0,0) \tag{4.6}$$

To do this, we choose a starting point

$$\mathbf{x}^0 = (x^0, y^0) \tag{4.7}$$

and two stepsizes in each coordinate direction

$$H = (h_1, h_2) \tag{4.8}$$

in such a way that the corresponding constructed box (see figure 5) forms a domain in which the method will attempt to locate and compute a root of the system $\mathsf{F}(\mathbf{x}) = \Theta$, which is a fixed point of the mapping T^P.
Suppose now that a fixed point x_1^* has been computed within a predetermined accuracy ϵ such that

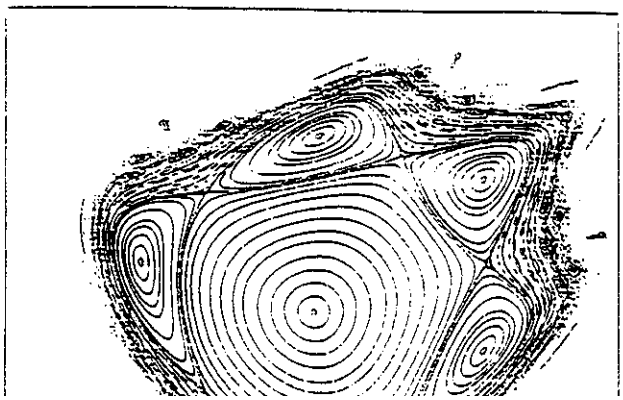
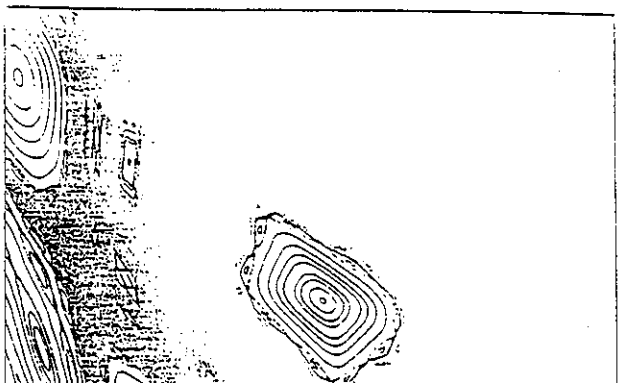
$$\|\mathsf{T}^P(\mathbf{x}_1^*) - \mathbf{x}_1^*\| \leq \epsilon. \tag{4.9}$$

Then, in order to compute all the other fixed points $\mathbf{x}_i^*$, $i = 2, \ldots, P$ with the same period and accuracy ϵ we iterate the mapping T as follows: first we obtain an approximation of the next fixed point $\mathbf{x}_2^*$ given by $\mathsf{T}(\mathbf{x}_1^*)$ and check if the following relationship is fulfilled:

$$\|\mathsf{T}^P(\mathsf{T}(\mathbf{x}_1^*)) - \mathsf{T}(\mathbf{x}_1^*)\| \leq \epsilon. \tag{4.10}$$

If so, then we choose $\mathbf{x}_2^* = \mathsf{T}(\mathbf{x}_1^*)$; otherwise we choose $\mathsf{T}(\mathbf{x}_1^*)$ as initial guess for $\mathbf{x}_2^*$ and refine the result until the desired accuracy is reached. The above procedure is applied repeatedly until the last fixed point x_P^* is obtained.
Let us now illustrate the above procedure on Hénon's mapping for a given value of α and various periods P. Thus, taking e.g. $\alpha = cos^{-1}(0.24)$ we can see in the phase plot of Hénon's mapping (see figure 7)
that there is a chain of five big islands around the origin. So, in this case, we can search for five elliptic and five hyperbolic fixed points of period $P = 5$ and rotation number $\sigma = 1/5$. We could also choose, of course, other values of α. The reason for this choice is that it produces a large region of stability around the origin and thus may be of interest to applications in beam dynamics.
Now, in order to find the elliptic fixed points of this period, we choose one island, include it into a box by taking appropriate values for $\mathbf{x}^0$ and H,

$$\mathbf{x}^0 = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ -0.3 \end{pmatrix}, \qquad H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix} \tag{4.11}$$

and apply the bisection method with accuracy $\epsilon = 10^{-16}$.
In this case, we compute the following fixed point utilizing around 6.5 msec of CPU time on the CERN VAX 9000-410 system:
$x_1^5 = 0.5672405470221847$ $\qquad$ $y_1^5 = -0.1223202134278941$

Of course, we can easily verify that this point is indeed an elliptic fixed point by applying the traditional technique of checking the eigenvalues of the linearized mapping[22]
Now, we can proceed with the computation of the rest of the elliptic fixed points of period five. To do this we can either apply the bisection method or iterate the mapping, since we have computed one of them within the predetermined accuracy of $\epsilon = 10^{-16}$. We prefer to do the latter and successively compute the following points
$x_2^5 = 0.5672405470221847$ $\qquad$ $y_2^5 = 0.4440820516139216$
$x_3^5 = 0.0173925844399303$ $\qquad$ $y_3^5 = 0.5800185952239573$